Online Appendix for Complementary Bayesian Method of Moments Strategies by A. Ronald Gallant

www.aronaldg.org/papers/cb_online.pdf

Asymptotic Normality

The probability space on which $Z(x,\theta)$ is defined has density $p^o(x\,|\,\theta)p^o(\theta)$, which is a mixture. To draw, one draws θ from $p^o(\theta)$ and then x from the conditional density $p^o(x\,|\,\theta)$. Fix such a θ draw. For data x generated according to $p^o(x\,|\,\theta)$ from that draw, define

$$F_n(z|\theta) = \int I[Z(x,\theta) \le z]\, p^o(x|\theta) dx.$$

Asymptotic normality for that θ is defined as

$$\lim_{n\to\infty} F_n(z\,|\,\theta) = \Phi(z) \tag{34}$$

at every continuity point of the right hand side, which is everywhere for the normal $\Phi(z)$. Conditions that imply asymptotic normality for cross-sectional data are given by Theorem 9 of Gallant(1987, p. 211) and for time series data by Theorem 10 Gallant(1987, p. 581).

To emphasize the dependence on sample size, write $\Psi_n(z)$ for the distribution of Z, and note that

$$\Psi_n(z) = \int F_n(z|\theta)\, p^o(\theta)\, d\theta. \tag{35}$$

The dominated convergence theorem (Royden and Fitzpatrick, 2010, p. 88), (34) implies

$$\lim_{n\to\infty} \Psi_n(z) = \int \lim_{n\to\infty} F_n(z|\theta)\, p^o(\theta)\, d\theta = \Phi(z).$$

Thus, asymptotic normality for any draw from $p^o(\theta)$ is enough to imply asymptotic normality of $Z(x,\theta)$ on the probability space $(\mathcal{X}\times\Theta, \mathcal{C}^o, P^o)$; asymptotic normality uniform over Θ is not required.[14]

Denote the density that corresponds to Ψ_n by ψ_n. To obtain $\lim_{n\to\infty}\psi_n(z) = \phi(z)$, $\psi_n(z)$ must be bounded and asymptotically equicontinuous (Sweeting, 1986). Uniformity is not required.

[14] I am indebted to Shengbo Zhu for pointing this out to me.

Additional References

Royden, H. L., and P. M. Fitzpatrick (2010), *Real Analysis, Fourth Edition*, Saddle River, NJ: Prentice-Hall.

Sweeting, Trevor J. (1986), "On a converse to Scheffe's theorem," *The Annals of Statistics* 14, 1252–1256.

Table A1. Exchange Economy Prior

	Normal Factor		Support Factor		Range in Simulation	
Parameter	Mean	Sdev	Left	Right	Left	Right
α	0.95	0.01	-0.99	0.99	0.914	0.988
σ	0.2	0.01	0.01	100.0	0.010	0.058
β	0.95	0.01	0.8	0.99	0.908	0.987
γ	12.5	2.0	0.0	100.0	4.719	20.10

For parameter $\rho = (\alpha, \sigma, \beta, \gamma)$ the prior has the form $\prod_{i=1}^{4} n(\rho_i, \text{Mean}_i, \text{Sdev}_i) I(\text{Left}_i < \rho_i < \text{Right}_i)$, for the values shown in the columns labeled Normal Factor and Support Factor, respectively. The effective support of the prior as determined from a simulation of size 10,000 is shown in the columns labeled Range in Simulation. At $\rho = (0.95, 0.02, 0.95, 12.5)$, in a simulation of size 200,000, the log endowment has mean 0.00000 and standard deviation 0.02024, the geometric stock return has mean 0.04490 and standard deviation 0.14332, the geometric risk free rate has mean 0.02019 and standard deviation 0.04032.

Table A2. Exchange Economy Likelihood and Adjustment, $n = 50$

Log Likelihood

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	-95.45	-4295.71	-70.45	-14.64	-6.03	-4.13	-3.86	-4.04	-4.29	-4.49	-4.62
10.45	-69.96	-3823.10	-51.70	-11.02	-5.10	-3.96	-3.95	-4.19	-4.45	-4.63	-4.73
20.40	-49.85	-3262.43	-36.57	-8.26	-4.47	-3.93	-4.09	-4.37	-4.61	-4.77	-4.85
30.35	-34.66	-2619.06	-24.70	-6.27	-4.12	-4.00	-4.27	-4.56	-4.78	-4.91	-4.96
40.30	-23.73	-1919.42	-15.77	-4.96	-4.00	-4.16	-4.49	-4.77	-4.96	-5.06	-5.07
50.25	-16.17	-1219.97	-9.53	-4.24	-4.08	-4.40	-4.74	-4.99	-5.14	-5.20	-5.19
60.20	-10.99	-607.12	-5.71	-4.03	-4.31	-4.70	-5.01	-5.21	-5.32	-5.34	-5.30
70.15	-7.45	-179.24	-4.06	-4.25	-4.69	-5.04	-5.30	-5.45	-5.51	-5.49	-5.41
80.10	-5.38	-13.38	-4.26	-4.81	-5.17	-5.43	-5.60	-5.69	-5.69	-5.63	-5.52
90.05	-4.97	-62.53	-5.87	-5.65	-5.74	-5.85	-5.92	-5.93	-5.88	-5.77	-5.63
100.00	-6.41	-361.68	-8.48	-6.69	-6.37	-6.29	-6.24	-6.17	-6.06	-5.91	-5.73

Log Adjustment

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	1.78	1.78	1.78	1.86	2.04	2.24	2.44	2.60	2.74	2.84	2.92
10.45	1.78	1.78	1.78	1.90	2.10	2.31	2.50	2.66	2.78	2.88	2.95
20.40	1.78	1.78	1.79	1.95	2.18	2.39	2.56	2.71	2.82	2.91	2.97
30.35	1.79	1.78	1.81	2.02	2.26	2.46	2.63	2.76	2.87	2.94	3.00
40.30	1.82	1.78	1.85	2.11	2.35	2.54	2.69	2.81	2.91	2.98	3.02
50.25	1.87	1.78	1.92	2.22	2.45	2.62	2.76	2.87	2.95	3.01	3.05
60.20	1.96	1.78	2.05	2.36	2.55	2.70	2.82	2.92	2.99	3.04	3.07
70.15	2.13	1.78	2.25	2.51	2.67	2.79	2.89	2.97	3.03	3.07	3.09
80.10	2.40	1.92	2.53	2.67	2.78	2.87	2.95	3.01	3.06	3.10	3.12
90.05	2.78	6.78	2.90	2.85	2.90	2.96	3.01	3.06	3.10	3.12	3.14
100.00	3.24	13.16	3.33	3.05	3.02	3.04	3.08	3.11	3.13	3.15	3.16

For sample size $n = 50$, ρ from the prior shown in Table A1 was sampled with $N = 1000$ repetitions. For each repetition, x with columns given by (19) was simulated from the exchange economy of Section 4. For $\theta = (\beta, \gamma)$ as shown, the likelihood is $p^*(x \,|, \theta) = \phi[Z(x, \theta)]$ with Z given by (14) and ϕ the standard normal density. The adjustment is given by (17). Shown are values for that ρ among the $N = 1000$ repetitions for which the difference between the maximum and minimum adjustment was largest. That value is $\rho = (0.9415, 0.01103, 0.9431, 17.89)$. Under the loose prior, the range of the MCMC draws reported in Table A5 were $0.8006 \leq \beta \leq 0.9899$, $0.07502 \leq \gamma \leq 84.46$ with adjustment and $0.8007 \leq \beta \leq 0.9898$, $0.01369 \leq \gamma \leq 82.19$ without.

Table A3. Exchange Economy Likelihood and Adjustment, $n = 100$

Log Likelihood

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	-211.99	-16640.04	-170.40	-39.47	-15.89	-8.78	-6.06	-4.92	-4.50	-4.45	-4.61
10.45	-155.11	-13398.32	-126.74	-29.62	-12.51	-7.39	-5.46	-4.70	-4.47	-4.53	-4.75
20.40	-109.32	-10306.79	-91.12	-21.81	-9.87	-6.33	-5.03	-4.58	-4.51	-4.66	-4.92
30.35	-73.48	-7474.01	-62.69	-15.78	-7.88	-5.56	-4.76	-4.55	-4.61	-4.82	-5.12
40.30	-46.57	-5002.77	-40.72	-11.29	-6.43	-5.05	-4.63	-4.61	-4.76	-5.02	-5.33
50.25	-27.68	-2978.95	-24.59	-8.12	-5.46	-4.76	-4.64	-4.74	-4.97	-5.25	-5.56
60.20	-15.73	-1462.56	-13.67	-6.07	-4.91	-4.69	-4.76	-4.95	-5.21	-5.50	-5.80
70.15	-9.34	-484.22	-7.26	-4.97	-4.72	-4.80	-4.98	-5.22	-5.49	-5.78	-6.06
80.10	-7.08	-58.66	-4.61	-4.70	-4.88	-5.08	-5.30	-5.55	-5.81	-6.07	-6.33
90.05	-8.10	-51.30	-5.04	-5.16	-5.33	-5.52	-5.72	-5.93	-6.16	-6.39	-6.60
100.00	-12.42	-508.94	-8.15	-6.29	-6.07	-6.10	-6.21	-6.36	-6.54	-6.72	-6.89

Log Adjustment

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	1.78	1.78	1.78	1.79	1.86	2.01	2.21	2.42	2.63	2.81	2.95
10.45	1.78	1.78	1.78	1.80	1.90	2.08	2.29	2.50	2.70	2.87	3.00
20.40	1.78	1.78	1.78	1.82	1.96	2.15	2.37	2.59	2.78	2.93	3.06
30.35	1.78	1.78	1.78	1.86	2.03	2.24	2.47	2.67	2.85	2.99	3.11
40.30	1.79	1.78	1.79	1.92	2.12	2.35	2.56	2.76	2.92	3.06	3.16
50.25	1.81	1.78	1.81	2.00	2.23	2.46	2.66	2.85	3.00	3.12	3.20
60.20	1.87	1.78	1.87	2.13	2.37	2.58	2.77	2.93	3.07	3.17	3.25
70.15	2.05	1.78	2.01	2.30	2.52	2.71	2.88	3.02	3.14	3.23	3.30
80.10	2.43	1.78	2.30	2.51	2.68	2.84	2.98	3.11	3.21	3.29	3.34
90.05	3.04	6.38	2.75	2.77	2.87	2.98	3.09	3.20	3.28	3.34	3.38
100.00	3.84	15.45	3.37	3.06	3.06	3.12	3.20	3.28	3.35	3.39	3.42

As for Table A2 but with sample size $n = 100$, $\rho = (0.9410, 0.01000, 0.9542, 15.00)$, and range $0.0.8105 \leq \beta \leq 0.9896$, $0.004808 \leq \gamma \leq 85.11$ with adjustment and $0.8002 \leq \beta \leq 0.9899$, $0.9028 \leq \gamma \leq 78.52$ without.

Table A4. Exchange Economy Likelihood and Adjustment, $n = 1000$

Log Likelihood

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	-1726.42	-113176.30	-1334.75	-282.69	-92.93	-36.92	-18.53	-14.47	-16.73	-21.69	-27.40
10.45	-1216.58	-80780.43	-949.73	-196.16	-63.81	-26.35	-15.65	-15.17	-19.10	-24.74	-30.57
20.40	-816.54	-54890.15	-643.10	-129.38	-42.29	-19.39	-14.68	-16.94	-22.07	-28.11	-33.89
30.35	-514.72	-34764.91	-406.01	-79.84	-27.40	-15.59	-15.40	-19.64	-25.57	-31.75	-37.34
40.30	-299.62	-19758.34	-230.73	-45.29	-18.29	-14.58	-17.62	-23.18	-29.51	-35.63	-40.88
50.25	-158.29	-9316.57	-110.38	-23.74	-14.24	-16.02	-21.14	-27.43	-33.85	-39.70	-44.51
60.20	-76.30	-2996.26	-38.55	-13.50	-14.65	-19.63	-25.83	-32.32	-38.52	-43.93	-48.20
70.15	-40.56	-419.18	-9.06	-13.14	-19.00	-25.18	-31.54	-37.76	-43.47	-48.29	-51.93
80.10	-42.62	-151.26	-16.16	-21.46	-26.85	-32.44	-38.16	-43.69	-48.68	-52.77	-55.70
90.05	-79.04	-2276.10	-54.96	-37.47	-37.83	-41.23	-45.57	-50.03	-54.09	-57.33	-59.49
100.00	-149.07	-7531.50	-121.60	-60.35	-51.59	-51.38	-53.69	-56.74	-59.67	-61.96	-63.29

Log Adjustment

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	1.78	1.78	1.78	1.78	1.78	1.81	2.06	2.71	3.57	4.37	5.00
10.45	1.78	1.78	1.78	1.78	1.78	1.87	2.28	3.08	3.95	4.69	5.26
20.40	1.78	1.78	1.78	1.78	1.80	1.99	2.61	3.49	4.32	5.00	5.51
30.35	1.78	1.78	1.78	1.78	1.85	2.23	3.03	3.92	4.69	5.30	5.75
40.30	1.78	1.78	1.78	1.79	1.99	2.62	3.51	4.35	5.05	5.59	5.99
50.25	1.78	1.78	1.78	1.85	2.31	3.15	4.02	4.78	5.40	5.87	6.21
60.20	1.78	1.78	1.79	2.12	2.89	3.76	4.55	5.21	5.74	6.14	6.42
70.15	2.00	1.78	2.08	2.86	3.67	4.41	5.07	5.62	6.06	6.40	6.63
80.10	3.82	9.54	3.82	4.05	4.54	5.08	5.58	6.02	6.38	6.65	6.83
90.05	6.84	30.26	6.53	5.40	5.44	5.73	6.08	6.41	6.68	6.89	7.02
100.00	9.87	inf	9.25	6.77	6.32	6.37	6.56	6.78	6.98	7.12	7.20

As for Table A2 but with sample size $n = 1000$, $\rho = (0.9377, 0.01009, 0.9424, 17.84)$, and range $0.8909 \leq \beta \leq 0.9496$, $12.92 \leq \gamma \leq 43.86$ with adjustment and $0.8539 \leq \beta \leq 0.9383$, $12.86 \leq \gamma \leq 40.35$ without.

Table A5. Exchange Economy Estimates

	No Adjustment			Adjustment		
Parameter	Mean	Mode	Sdev	Mean	Mode	Sdev
			$n = 50$, tight prior			
β	0.9530	0.9550	0.00513	0.9540	0.9559	0.00510
γ	13.650	12.365	1.4849	13.603	12.233	1.4800
			$n = 100$, tight prior			
β	0.9595	0.9603	0.00422	0.9604	0.9611	0.00428
γ	13.603	12.233	1.4800	13.994	12.994	1.3948
			$n = 1000$, tight prior			
β	0.9312	0.9315	0.00242	0.9442	0.9445	0.00169
γ	14.995	14.673	0.8594	14.900	14.539	0.7894
			$n = 50$, loose prior			
β	0.9177	0.9561	0.03884	0.9228	0.9569	0.03904
γ	31.145	12.025	12.893	31.725	11.939	14.016
			$n = 100$, loose prior			
β	0.9416	0.9614	0.03017	0.9449	0.9623	0.03022
γ	33.272	13.301	13.402	34.613	13.189	14.213
			$n = 1000$, loose prior			
β	0.9280	0.9296	0.00493	0.9433	0.9441	0.00282
γ	16.539	15.414	2.0841	15.870	14.912	1.8355

The x with columns given by (19) are a simulation from the exchange economy of Section 4 with with ρ set to the values shown in Tables A2 through A4. Without adjustment the likelihood is $p^*(x\,|,\theta) = \phi[Z(x,\theta)]$ with Z given by (14) and ϕ denoting the standard normal density. With adjustment the likelihood is $p^*(u\,|,\theta) = \text{adj}(u,\theta)\phi[Z(u,\theta)]$ with u defined by (15), Z defined by (16), and adj defined by (17). $\theta = (\beta,\gamma)$, a subvector of ρ. The tight prior for θ is the last two rows shown in Table A1. The loose prior is the same with standard deviations multiplied by ten. Estimates are from an MCMC chain (Gamerman and Lopes, 2006) of length $N = 200000$ collected past the point where transients have died out. The proposal is move-one-at-a-time random walk. Mean and standard deviation are computed with a stride of 100; mode is that over the entire chain.

Table A6. Discounted Cash Flow Prior

	Normal Factor		Support Factor		Range in Simulation	
Parameter	Mean	Sdev	Left	Right	Left	Right
$b_0(1)$	-1.6040	0.0143	$-\infty$	∞	-1.6506	-1.5578
$b_0(2)$	0.8663	0.0234	$-\infty$	∞	0.7835	0.9388
$b_0(3)$	-0.9020	0.0376	$-\infty$	∞	-1.0379	-0.7732
$B(1,1)$	-0.0003	0.0006	$-\infty$	∞	-0.0021	0.0016
$B(2,1)$	0.0186	0.0093	$-\infty$	∞	-0.0128	0.0501
$B(3,1)$	-0.0226	0.0103	$-\infty$	∞	-0.0558	0.0158
$B(1,2)$	-0.0020	0.0005	$-\infty$	∞	-0.0036	-0.0002
$B(2,2)$	0.6730	0.0095	$-\infty$	∞	0.6450	0.7083
$B(3,2)$	0.3457	0.0126	$-\infty$	∞	0.2979	0.3921
$B(1,3)$	-0.0017	0.0061	$-\infty$	∞	-0.0210	0.0183
$B(2,3)$	-0.1782	0.0941	$-\infty$	∞	-0.4941	0.1746
$B(3,3)$	0.3144	0.1018	$-\infty$	∞	-0.0225	0.6647
$R(1,1)$	1.7636	0.0083	$-\infty$	∞	1.7313	1.7941
$R(1,2)$	-0.1093	0.0402	$-\infty$	∞	-0.2404	0.0450
$R(2,2)$	0.0314	0.0027	$-\infty$	∞	0.0214	0.0422
$R(1,3)$	-0.2343	0.0336	$-\infty$	∞	-0.3386	-0.1280
$R(2,3)$	-0.1373	0.0121	$-\infty$	∞	-0.1867	-0.0929
$R(3,3)$	0.1575	0.0128	$-\infty$	∞	0.1141	0.2076
β	0.9532	0.0547	0.8	0.99	0.8003	0.9899
γ	24.5030	28.2425	0	100	0.0479	99.5641

The prior to determine the distribution of Z (e.g., Figure A2), is over $\rho = (b_0, B, R, \beta, \gamma)$. The prior is multivariate normal with means and standard errors as shown here and correlations as shown in Table A7 subject to support conditions shown here. The VAR $y_t = b_0 + By_{t-1} + Ru$ with u independent normal determines the the distribution of $y = \log$ (marginal rate of substitution, GDP/corporate profits, corporate profits). For each $\tilde{\rho}$ drawn from the prior, a realization $\{y_t\}_{t=1}^{n}$ of the VAR with initial condition y_0 for 2015 is computed. From each realization, the return on corporate profits conditional on y_t is computed analytically from the drawn $(\tilde{b}_0, \tilde{B}, \tilde{R})$ and the marginal rate of substitution is converted to consumption growth using CRRA utility with drawn $(\tilde{\beta}, \tilde{\gamma})$. The resulting bivariate geometric returns to corporate profits and log consumption growth series is used to compute the Z corresponding to $\tilde{\rho}$. In a simulation of size 5,000 with ρ set to the prior mean shown here, log consumption growth has mean 0.0020 and standard deviation 0.1441, the geometric return on corporate profits has mean 0.06310 and standard deviation 0.07255. To estimate $\theta = (\beta, \gamma)$ from bivariate returns and consumption growth data using the method described in Section 2 the prior is bivariate normal with means and standard deviations as shown here for (β, γ), zero correlation, and support conditions $0.8 \leq \beta \leq 0.99$ and $0 \leq \gamma \leq 100$.

Table A7. Discounted Cash Flow Prior Correlations

$b_0(1)$																			
	$b_0(2)$																		
	−○	$b_0(3)$																	
			$B(1,1)$																
	+○	−○		$B(2,1)$															
	−○	+○		−●	$B(3,1)$														
			−•			$B(1,2)$													
−○	−•						$B(2,2)$												
		−●					−○	$B(3,2)$											
						+○			$B(1,3)$										
		−○		+○	−○					$B(2,3)$									
		+○		−○	+○					−•	$B(3,3)$								
−•												$R(1,1)$							
	+○						−○	+○		+○	−○	+•	$R(1,2)$						
														$R(2,2)$					
+○															$R(1,3)$				
		−○		+○	−○					+○	−○					$R(2,3)$			
		+○		−○	+○					−○	+○					−●	$R(3,3)$		
																		β	
																			γ

The prior over $\rho = (b_0, B, R, \beta, \gamma)$ that is used to determine the distribution of Z is multivariate normal with means and standard errors as shown in Table A6 and correlation matrix as shown here. A blank indicates a correlation of less than 0.25 in absolute value. A ○ indicates a correlation with absolute value between 0.25 and 0.5 with sign as shown. Similarly a • for between 0.5 and 0.75 and ● for larger than 0.75.

Table A8. Discounted Corporate Profits Likelihood, $n = 50$

Log Likelihood

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	-237.32	-14.67	-5.56	-5.96	-6.29	-6.35	-6.30	-3.17	-3.17	-3.17	-3.17
10.45	-202.61	-14.00	-5.54	-5.96	-6.29	-6.34	-6.28	-3.17	-3.17	-3.17	-3.17
20.40	-171.87	-13.39	-5.52	-5.96	-6.28	-6.32	-6.27	-3.17	-3.17	-3.17	-3.17
30.35	-144.75	-12.82	-5.51	-5.96	-6.27	-6.31	-6.25	-3.17	-3.17	-3.17	-3.17
40.30	-120.93	-12.29	-5.49	-5.96	-6.27	-6.30	-6.24	-3.17	-3.17	-3.17	-3.17
50.25	-100.10	-11.80	-5.48	-5.96	-6.26	-6.29	-6.23	-3.17	-3.17	-3.17	-3.17
60.20	-82.00	-11.34	-5.46	-5.96	-6.25	-6.28	-6.21	-3.17	-3.17	-3.17	-3.17
70.15	-66.39	-10.92	-5.45	-5.96	-6.24	-6.27	-6.20	-3.17	-3.17	-3.17	-3.17
80.10	-53.02	-10.52	-5.44	-5.96	-6.24	-6.26	-6.18	-3.17	-3.17	-3.17	-3.17
90.05	-41.69	-10.15	-5.43	-5.96	-6.23	-6.24	-6.17	-3.17	-3.17	-3.17	-3.17
100.00	-32.19	-9.81	-5.42	-5.96	-6.22	-6.23	-6.15	-3.17	-3.17	-3.17	-3.17

Log Transformed Likelihood

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	-14.67	-7.39	-5.77	-5.96	-6.32	-6.37	-6.31	-3.21	-3.21	-3.21	-3.21
10.45	-14.18	-7.26	-5.74	-5.97	-6.31	-6.36	-6.30	-3.21	-3.21	-3.21	-3.21
20.40	-13.14	-7.15	-5.70	-5.97	-6.31	-6.35	-6.29	-3.21	-3.21	-3.21	-3.21
30.35	-13.24	-7.08	-5.66	-5.97	-6.30	-6.34	-6.27	-3.21	-3.21	-3.21	-3.21
40.30	-12.14	-7.03	-5.62	-5.97	-6.29	-6.32	-6.26	-3.21	-3.21	-3.21	-3.21
50.25	-11.09	-7.01	-5.59	-5.97	-6.28	-6.31	-6.24	-3.21	-3.21	-3.21	-3.21
60.20	-9.52	-6.99	-5.56	-5.97	-6.28	-6.30	-6.23	-3.21	-3.21	-3.21	-3.21
70.15	-9.94	-6.96	-5.53	-5.97	-6.27	-6.29	-6.22	-3.21	-3.21	-3.21	-3.21
80.10	-7.76	-6.92	-5.50	-5.97	-6.26	-6.28	-6.20	-3.21	-3.21	-3.21	-3.21
90.05	-7.89	-6.86	-5.48	-5.97	-6.25	-6.26	-6.19	-3.21	-3.21	-3.21	-3.21
100.00	-8.30	-6.79	-5.46	-5.97	-6.25	-6.25	-6.17	-3.21	-3.21	-3.21	-3.21

For sample size $n = 50$, ρ from the prior shown in Table A6 was sampled with $N = 1000$ repetitions. For each repetition, x with columns given by (10) was computed from simulated cash flows as described in Section 5. For $\theta = (\beta, \gamma)$ as shown, the likelihood is $p^*(x\,|, \theta) = \psi[Z(x, \theta)]$ with Z given by (14) and ψ the standard normal density for the panel labeled "Log Likelihood" and (29) for the panel labeled "Log Transformed Likelihood"; Shown are values for that ρ among the $N = 1000$ repetitions for which the absolute value of the difference between the likelihoods was largest. For this draw $\theta = (\beta, \gamma) = (0.8991, 6.359)$. The range of the MCMC draws reported in Table A11 were $0.8008 \leq \beta \leq 0.9900$, $0.05273 \leq \gamma \leq 99.68$ with adjustment and $0.8049 \leq \beta \leq 0.9900$, $0.40184 \leq \gamma \leq 94.57$ without.

Table A9. Discounted Corporate Profits Likelihood, $n = 100$

Log Likelihood

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	-335.85	-4.20	-4.03	-4.52	-4.54	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
10.45	-279.67	-4.07	-4.05	-4.52	-4.54	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
20.40	-230.82	-3.96	-4.06	-4.53	-4.54	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
30.35	-188.51	-3.86	-4.07	-4.53	-4.55	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
40.30	-152.08	-3.77	-4.08	-4.53	-4.55	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
50.25	-120.91	-3.69	-4.09	-4.53	-4.55	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
60.20	-94.47	-3.62	-4.11	-4.54	-4.55	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
70.15	-72.27	-3.55	-4.12	-4.54	-4.55	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
80.10	-53.88	-3.49	-4.13	-4.54	-4.55	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
90.05	-38.91	-3.44	-4.14	-4.54	-4.55	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17
100.00	-27.02	-3.40	-4.15	-4.54	-4.55	-4.50	-4.45	-4.41	-3.17	-3.17	-3.17

Log Transformed Likelihood

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	-9.47	-4.49	-4.02	-4.79	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
10.45	-8.82	-4.42	-4.03	-4.79	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
20.40	-8.31	-4.35	-4.04	-4.79	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
30.35	-7.64	-4.26	-4.04	-4.79	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
40.30	-8.37	-4.17	-4.05	-4.79	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
50.25	-9.19	-4.07	-4.06	-4.80	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
60.20	-7.95	-3.96	-4.07	-4.80	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
70.15	-6.63	-3.85	-4.07	-4.80	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
80.10	-6.27	-3.75	-4.08	-4.80	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
90.05	-7.73	-3.65	-4.09	-4.80	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33
100.00	-7.49	-3.55	-4.10	-4.80	-4.89	-4.84	-4.79	-4.75	-3.33	-3.33	-3.33

As for Table A8 but with sample size $n = 100$, $\theta = (\beta, \gamma) = (0.9133, 9.427)$. $0.8036 \leq \beta \leq 0.9900$, $0.4313 \leq \gamma \leq 94.32$ with adjustment and $0.8028 \leq \beta \leq 0.9900$, $0.2902 \leq \gamma \leq 94.53$ without.

Table A10. Discounted Corporate Profits Likelihood, $n = 1000$

Log Likelihood

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	-3370.20	-651.29	-90.64	-5.13	-5.45	-6.22	-5.87	-5.47	-5.17	-4.97	-4.83
10.45	-2654.69	-589.82	-82.32	-4.87	-5.54	-6.25	-5.88	-5.47	-5.17	-4.97	-4.83
20.40	-2053.04	-534.00	-74.76	-4.66	-5.63	-6.27	-5.88	-5.47	-5.17	-4.97	-4.83
30.35	-1553.27	-483.26	-67.87	-4.49	-5.71	-6.30	-5.89	-5.47	-5.17	-4.97	-4.83
40.30	-1144.87	-437.10	-61.61	-4.35	-5.80	-6.32	-5.89	-5.47	-5.17	-4.97	-4.83
50.25	-818.50	-395.08	-55.90	-4.25	-5.88	-6.34	-5.90	-5.47	-5.17	-4.97	-4.83
60.20	-565.54	-356.81	-50.70	-4.18	-5.96	-6.37	-5.90	-5.47	-5.18	-4.97	-4.83
70.15	-377.53	-321.93	-45.97	-4.13	-6.04	-6.39	-5.91	-5.47	-5.18	-4.97	-4.83
80.10	-245.61	-290.13	-41.65	-4.11	-6.12	-6.41	-5.91	-5.47	-5.18	-4.97	-4.83
90.05	-159.99	-261.14	-37.73	-4.10	-6.19	-6.43	-5.92	-5.48	-5.18	-4.97	-4.83
100.00	-110.14	-234.71	-34.15	-4.12	-6.27	-6.45	-5.92	-5.48	-5.18	-4.97	-4.83

Log Transformed Likelihood

γ/β	0.80	0.82	0.84	0.86	0.88	0.90	0.91	0.93	0.95	0.97	0.99
0.50	-299.60	-29.99	-39.02	-5.28	-8.55	-8.46	-7.62	-7.87	-7.29	-5.87	-7.12
10.45	-206.38	-30.87	-32.52	-5.00	-10.00	-8.30	-7.60	-7.88	-7.29	-5.87	-7.12
20.40	-131.38	-30.91	-30.32	-4.97	-8.32	-8.17	-7.59	-7.89	-7.30	-5.87	-7.12
30.35	-74.79	-29.45	-28.53	-4.67	-7.81	-8.05	-7.58	-7.90	-7.30	-5.87	-7.12
40.30	-36.72	-28.59	-26.99	-4.34	-7.54	-7.95	-7.57	-7.90	-7.31	-5.87	-7.12
50.25	-15.79	-27.87	-25.65	-4.33	-7.37	-7.86	-7.56	-7.91	-7.31	-5.87	-7.12
60.20	-11.52	-27.18	-24.49	-4.30	-7.11	-7.78	-7.55	-7.92	-7.31	-5.87	-7.12
70.15	-16.05	-26.50	-23.54	-4.19	-7.20	-7.70	-7.54	-7.93	-7.32	-5.87	-7.12
80.10	-31.39	-25.82	-22.87	-4.02	-8.62	-7.64	-7.53	-7.93	-7.32	-5.87	-7.12
90.05	-52.84	-25.13	-21.83	-3.92	-8.38	-7.57	-7.53	-7.94	-7.32	-5.87	-7.12
100.00	-72.31	-24.43	-19.40	-4.19	-7.98	-7.52	-7.52	-7.94	-7.32	-5.87	-7.12

As for Table A8 but with sample size $n = 1000$, $\theta = (\beta, \gamma) = (0.8832, 25.24)$. $0.8012 \leq \beta \leq 0.9900$, $4.635 \leq \gamma \leq 99.94$ with adjustment and $0.8046 \leq \beta \leq 0.9900$, $7.306 \leq \gamma \leq 99.27$ without.

Table A11. Discounted Cash Flow Estimates

	No Adjustment			Adjustment		
Parameter	Mean	Mode	IQR	Mean	Mode	IQR
			$n = 50$			
β	0.9328	0.9556	0.05590	0.9328	0.9559	0.05582
γ	39.595	29.902	17.422	42.082	31.859	18.922
			$n = 100$			
β	0.9311	0.9541	0.05653	0.9311	0.9538	0.05666
γ	30.151	25.574	8.6111	31.737	26.849	9.3107
			$n = 1000$			
β	0.9310	0.9556	0.05631	0.9305	0.9528	0.05669
γ	40.288	28.940	4.3800	37.489	25.680	5.7771
			$n = 50$, transformed Z			
β	0.9327	0.9558	0.05562	0.9327	0.9574	0.05606
γ	35.500	29.562	18.321	39.249	38.540	18.720
			$n = 100$, transformed Z			
β	0.9315	0.9539	0.05652	0.9308	0.9533	0.05677
γ	26.797	25.657	8.4857	28.833	29.564	8.3368
			$n = 1000$, transformed Z			
β	0.9327	0.9604	0.05570	0.9322	0.9567	0.05606
γ	32.624	27.330	3.0234	35.546	28.259	5.0233

The data x are a simulation of log corporate profit returns and log consumption growth as described in the legend of Table A6 with ρ set to the column labeled Mean in Table A6; specifically, $\theta = (\beta, \gamma) = (0.9532, 24.5030)$ for the subvector θ of ρ. Without adjustment the likelihood is $p^*(x \,|, \theta) = \phi[Z(x, \theta)]$ with Z given by (14) and ϕ denoting the standard normal density. With adjustment the likelihood is $p^*(u \,|, \theta) = \text{adj}(u, \theta)\phi[Z(u, \theta)]$ with u defined by (15), Z defined by (16), and adj defined by (17). Transformed Z are computed as described in the legend to Figure A3. Estimates are from an MCMC chain (Gamerman and Lopes, 2006) of length $N = 800000$ collected past the point where transients have died out. The proposal is move-one-at-a-time random walk. Mean and interquartile range (IQR), and mode are computed with a stride of one.

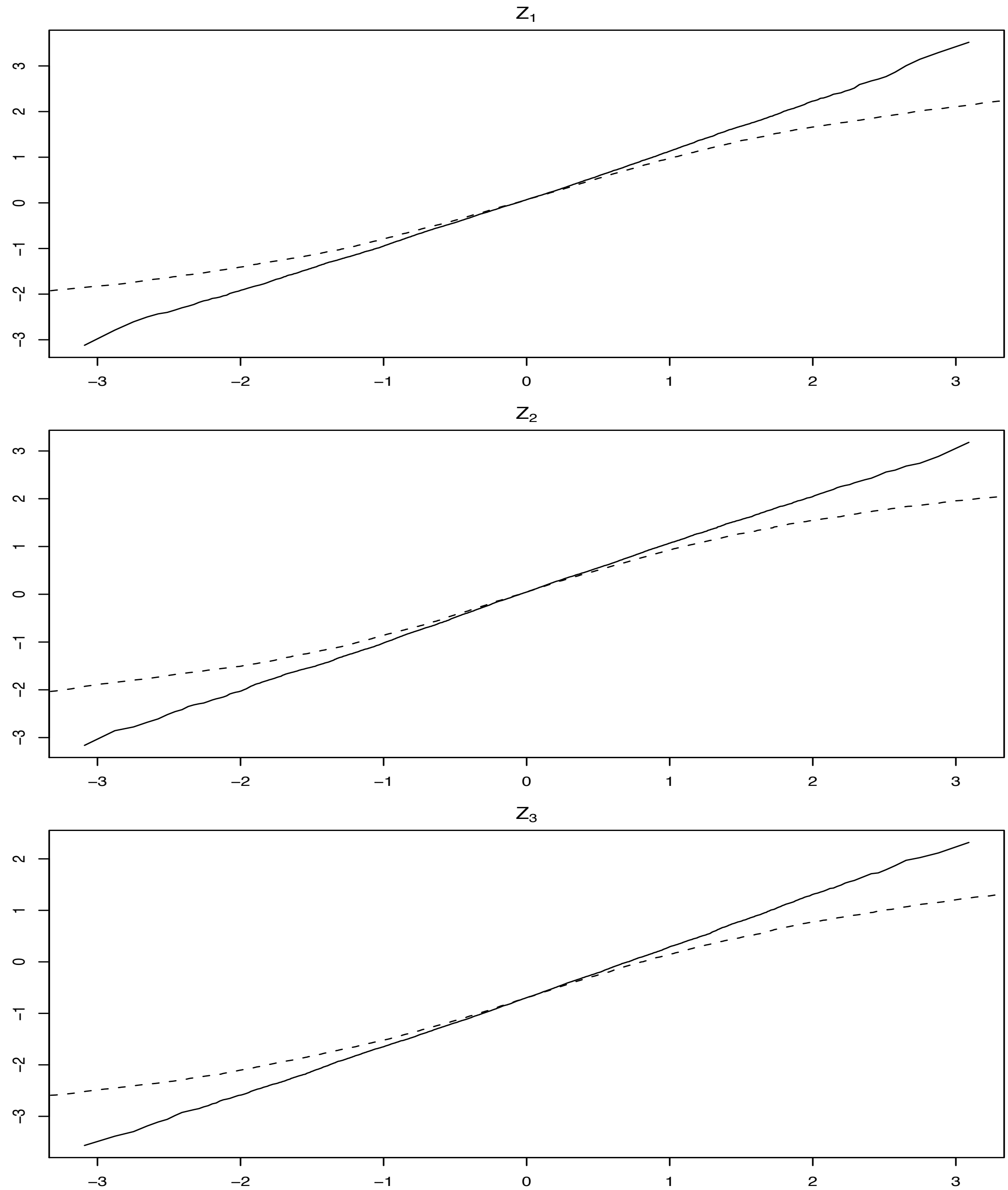

Figure A1. Exchange Economy Q-Q Plots. For the exchange economy of Section 4, $N = 10000$ values for $\rho = (\alpha, \sigma, \beta, \gamma)$ were drawn from the prior shown in Table A1. For each ρ, data x with columns given by (19) were computed for $t = 1, \ldots, n$, $n = 100$. For each x and subvector $\theta = (\beta, \gamma)$ of ρ the random variable $z = Z(x, \theta)$ defined by (14) was computed. From the $N = 10000$ values of z thus computed, quantiles at probabilities 0.001 through 0.999 at increments of 0.001 were computed for each of the elements z_1, z_2, and z_3 of z. Similarly for the normal and Student's t-distribution on 3 degrees freedom. Plotted are the z quantiles (vertical axis) against the normal quantiles (solid line) and the t-quantiles (dotted line).

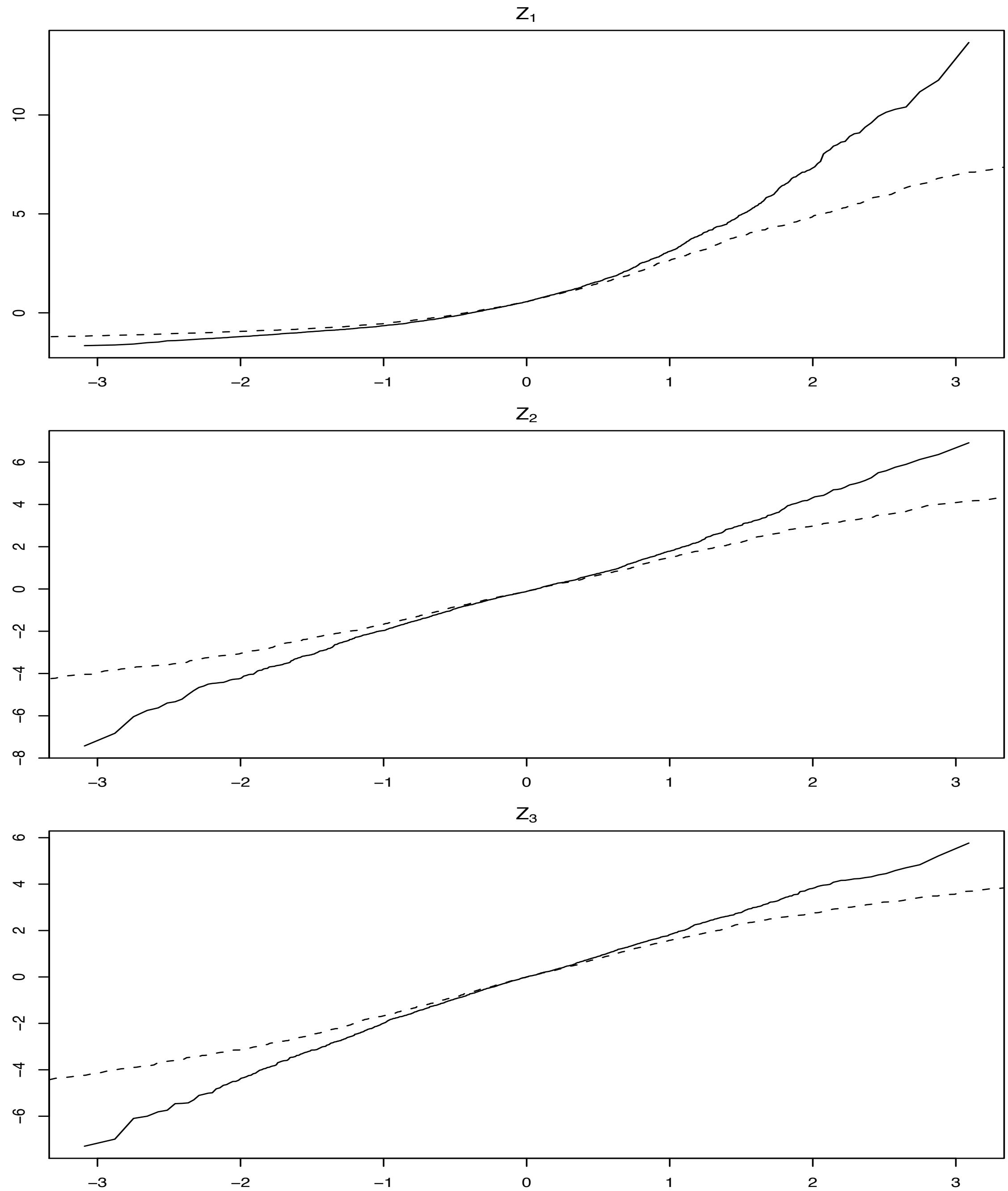

Figure A2. Discounted Cash Flow Raw Q-Q Plots. For the discounted cash flows setup of Section 5, $N = 2000$ values for $\rho = (b_0, B, R, \beta, \gamma)$ were drawn from the prior shown in Table A6. For each ρ, data $x = \{x_t\}_{t=1}^{n}$ were computed for $n = 50$ as described in the legend to Table A6. For each x and subvector $\theta = (\beta, \gamma)$ of ρ the random variable $z = Z(x, \theta)$ was computed as described in the legend to Table A6. From the $N = 2000$ values of z thus computed, quantiles at probabilities 0.001 through 0.999 at increments of 0.001 were computed for each of the elements z_1, z_2, and z_3 of z. Plotted are the z quantiles (vertical axis) against the normal quantiles (solid line) and the t-quantiles (dotted line).

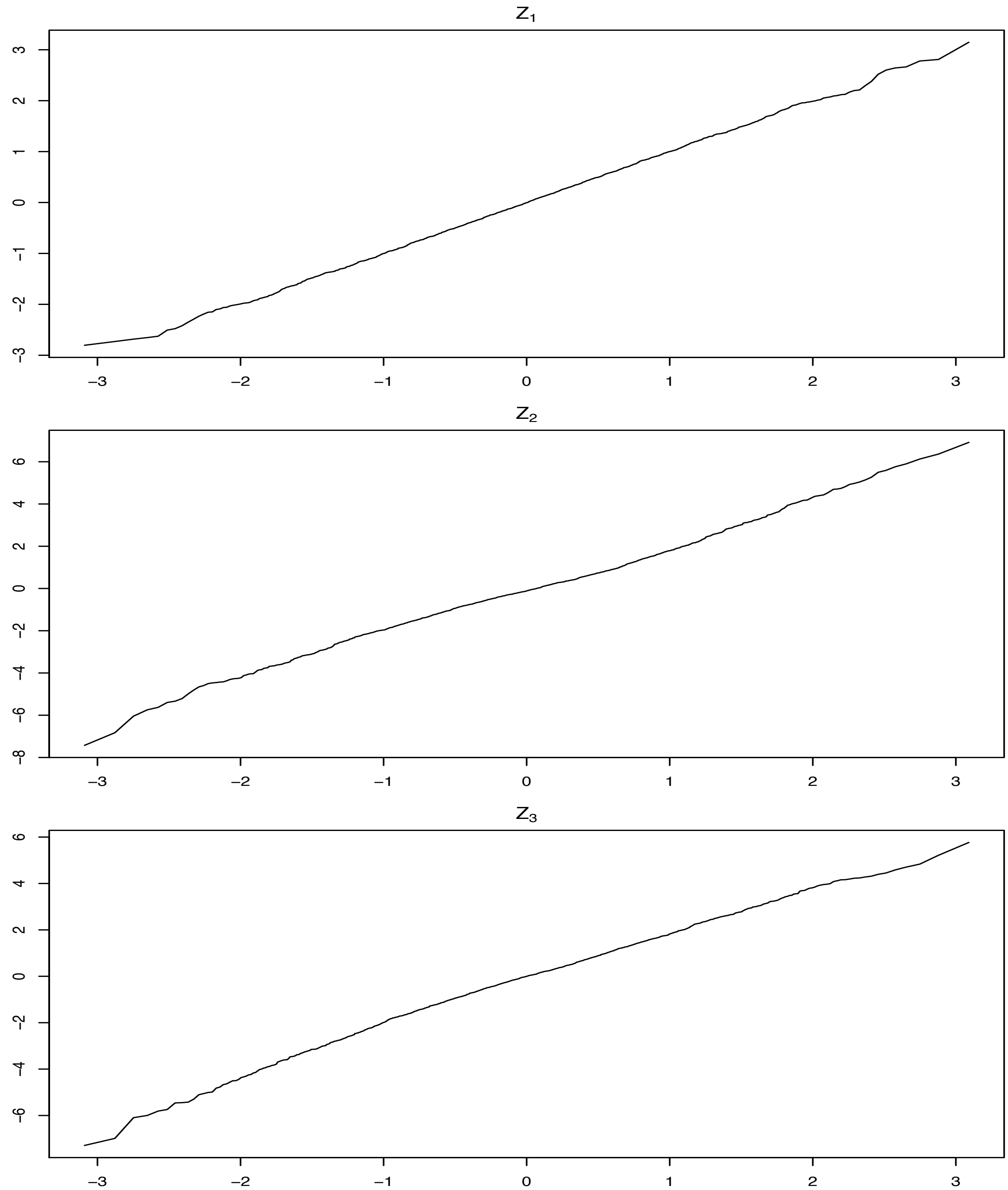

Figure A3. Discounted Cash Flow Transformed Q-Q Plots. For the discounted cash flows setup of Section 5 and draws from the prior shown in Table A6, the random variable $z = Z(x, \theta)$ was computed as described in the legend for Figure A2. Each $z = (z_1, z_2, z_3)$ was transformed to $\tilde{z} = \big(\Phi^{-1}[F_{z_1}(z_1)], z_2, z_3\big)$, where Φ is the distribution function of the standard normal and F_{z_1} is a natural cubic spline interpolation of the empirical distribution of z_1. The correlations among the $\tilde{z}$ are -0.0451, 0.0088, 0.0015 in the order (1,2), (1,3), (2,3). From the $N = 2000$ values of $\tilde{z}$ thus computed, quantiles at probabilities 0.001 through 0.999 at increments of 0.001 were computed for each of the elements of $\tilde{z}$. Plotted are the $\tilde{z}$ quantiles (vertical axis) against the normal quantiles.

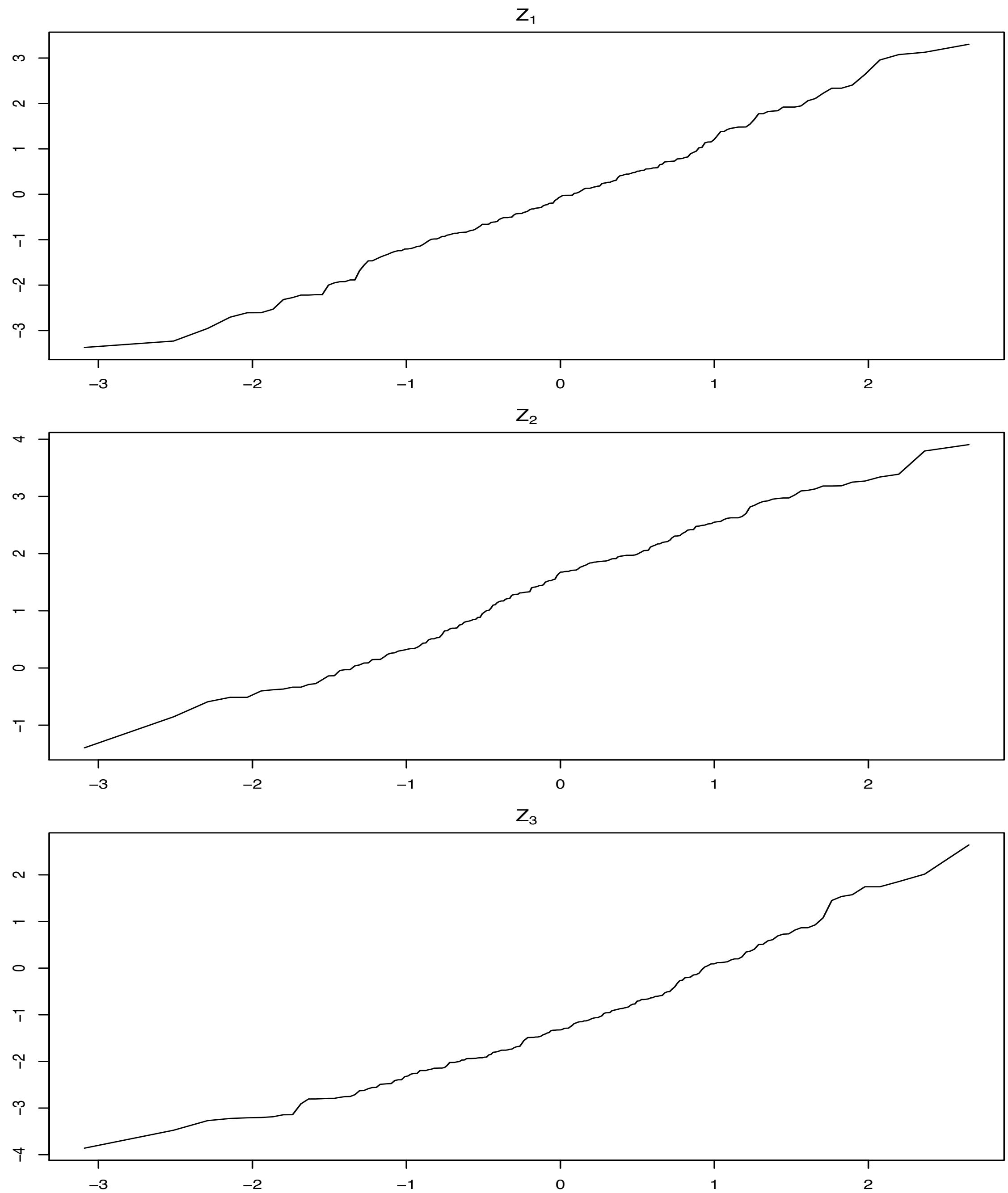

Figure A4. Complementary Q-Q Plots for Exchange Economy. The prior $p^o(\theta)$ given in the legend of Table A1 for the exchange economy of Section 4 was discretized by a quadrature rule to obtain a discrete prior $(\theta_i^o, p_i^o), i = 1, \ldots, 16$. Using the λ-prior method described in Section 6 with $p_\lambda^i(\rho, \theta)$ defined by (33), the discrete prior $(\rho_i^o, p_i^o), i = 1, \ldots, 16$, was determined for the likelihood $f_{SNP}(x, \rho)$. In the notation of Gallant and Tauchen (2017), the BIC determined SNP specification has parameters $L_u = 1,$, $K_z = 4$, and $I_z = 3$, with all other parameters zero. Sample size is $n = 500$; $\lambda = 10^{2.5} \approx 316$. $N = 1000$ values for p were drawn from the discrete prior (ρ_i^o, p_i^o). For each ρ drawn, data x of length $n = 500$ were drawn from $f_{SNP}(x, \rho)$. For each x, the random variable $z = Z(x, \theta)$ defined by (14) was computed. From the $N = 1000$ values of z thus computed, quantiles at probabilities 0.001 through 0.999 at increments of 0.005 were computed for each of the elements z_1, z_2, and z_3 of z. Similarly for the normal. Plotted are the z quantiles against the normal quantiles.

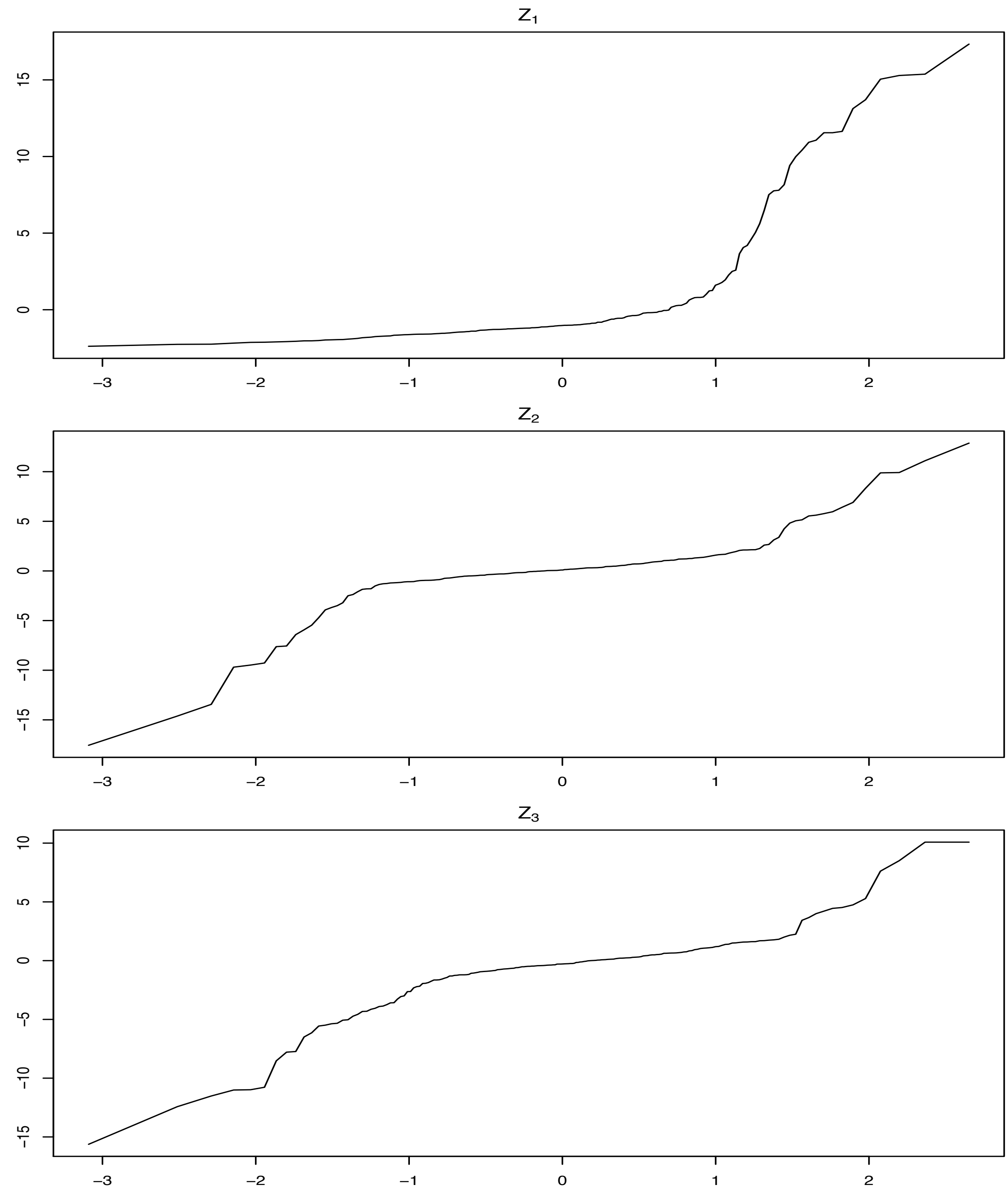

Figure A5. Complementary Q-Q Plots for Discounted Cash Flows. The prior $p^o(\theta)$ given in the legend of Table A6 for the discounted cash flows setup of Section 5 was discretized by a quadrature rule to obtain a discrete prior $(\theta_i^o, p_i^o), i = 1, \ldots, 16$. Using the λ-prior method described in Section 6 with $p_\lambda^i(\rho, \theta)$ defined by (33), the discrete prior $(\rho_i^o, p_i^o), i = 1, \ldots, 16$, was determined for the likelihood $f_{SNP}(x, \rho)$. In the notation of Gallant and Tauchen (2017), the BIC determined SNP specification has parameters $L_u = 1$, and $K_z = 4$, with all other parameters zero. Sample size is $n = 500$; $\lambda = 10^{1.25} \approx 1.25$. $N = 1000$ values for p were drawn from the discrete prior (ρ_i^o, p_i^o). For each ρ drawn, data x of length $n = 500$ were drawn from $f_{SNP}(x, \rho)$. For each x, the random variable $z = Z(x, \theta)$ defined by (14) was computed. From the $N = 1000$ values of z thus computed, quantiles at probabilities 0.001 through 0.999 at increments of 0.005 were computed for each of the elements z_1, z_2, and z_3 of z. Similarly for the normal. Plotted are the z quantiles against the normal quantiles.