

A UNIFORM CENTRAL LIMIT THEOREM
USEFUL IN NONLINEAR TIME SERIES REGRESSION

by

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The purpose of this report is to prove in detail a central limit theorem useful in nonlinear time series regression. The main ideas of the proof are due to Goebel (1974). The contribution here is to obtain a stronger conclusion than his Theorem 3 and correct some minor errors in his proof.

Lemma 1. Let

$$S_n(\theta) = Z_{kn}(\theta) + X_{kn}(\theta) \quad (k = 1, 2, \dots, n = 1, 2, \dots)$$

Assume that for every $\delta > 0$

$$\lim_{k \rightarrow \infty} P[|X_{kn}(\theta)| > \delta] = 0$$

uniformly in n and θ . Assume

$$\lim_{n \rightarrow \infty} P[Z_{kn}(\theta) \leq z] = N(z; 0, \sigma_k^2(\theta))$$

and

$$\lim_{k \rightarrow \infty} \sigma_k^2(\theta) = \tau^2(\theta)$$

uniformly in θ where $0 < l \leq \tau^2(\theta) \leq \mu < \infty$ for all θ . Then

$$\lim_{n \rightarrow \infty} P[S_n(\theta) \leq z] = N(z; 0, \tau^2(\theta))$$

uniformly in θ .

Proof: Given $\epsilon > 0$ there is a $\delta > 0$ depending on ϵ but not θ or n such that

$$N(z + \delta; 0, \tau^2(\theta)) < N(z; 0, \tau^2(\theta)) + \epsilon$$

because $\tau^2(\theta)$ is suitably bounded from above and below. There is a

k which depends on δ but not on θ or n such that

$$N(z + \delta; 0, \sigma_k^2(\theta)) < N(z + \delta; 0, \tau^2(\theta)) + \epsilon$$

$$P[|X_{kn}(\theta)| > \delta] < \epsilon$$

by hypothesis and the uniform convergence of $\sigma_k^2(\theta)$ to $\tau^2(\theta)$. Then there is an n^* depending on k and δ but not on θ such that for all $n > n^*$

$$P[Z_{kn}(\theta) \leq z + \delta] < N(z + \delta; 0, \sigma_k^2(\theta)) + \epsilon.$$

Consequently, given $\epsilon > 0$ there is an n^* which does not depend on θ such that

$$\begin{aligned} P[S_n(\theta) \leq z] &= P[Z_{kn}(\theta) + X_{kn}(\theta) \leq z] \\ &\leq P[Z_{kn}(\theta) \leq z + \delta, |X_{kn}(\theta)| \leq \delta] + P[|X_{kn}(\theta)| > \delta] \\ &< P[Z_{kn}(\theta) \leq z + \delta] + \epsilon \\ &< N(z + \delta; 0, \sigma_k^2(\theta)) + 2\epsilon \\ &< N(z + \delta; 0, \tau^2(\theta)) + 3\epsilon \\ &< N(z; 0, \tau^2(\theta)) + 4\epsilon \end{aligned}$$

for all $n > n^*$. Similar arguments can be used to show that

$$P[S_n(\theta) \leq z] > N(z; 0, \tau^2(\theta)) - 4\epsilon$$

for $n > n'$ where n' does not depend on θ . \square

Lemma 2. If $\sup_{\theta} c_t^2(\theta) < \infty$ for each t and

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n c_t^2(\theta) = \bar{c}(\theta)$$

uniformly in θ where $0 < \bar{l} \leq \bar{c}(\theta) \leq \bar{\mu} < \infty$ for all θ then

$$\lim_{n \rightarrow \infty} n^{-1} \sup_{1 \leq t \leq n} \sup_{\theta} c_t^2(\theta) = 0.$$

Proof: There is a sequence m_n such that

$$\sup_{1 \leq t \leq n} \sup_{\theta} c_t^2(\theta) = \sup_{\theta} c_{m_n}^2(\theta).$$

The lemma would be true trivially if m_n were bounded for all n so we assume the contrary. Given $\epsilon > 0$ there is an n^* which does not depend on θ such that for $n > n^*$ we have

$$\begin{aligned} & n^{-1} \sup_{1 \leq t \leq n} \sup_{\theta} c_t^2(\theta) \\ &= \sup_{\theta} \left[\left(\frac{m_n}{n} \right) \frac{1}{m_n} \sum_{t=1}^{m_n} c_t^2(\theta) - \left(\frac{m_n-1}{n} \right) \frac{1}{m_n-1} \sum_{t=1}^{m_n-1} c_t^2(\theta) \right] \\ &< \sup_{\theta} \left[\left(\frac{m_n}{n} \right) (\bar{c}(\theta) + \epsilon) - \left(\frac{m_n-1}{n} \right) (\bar{c}(\theta) - \epsilon) \right] \\ &= \sup_{\theta} \left[\bar{c}(\theta)/n + 2m_n \epsilon/n + \epsilon/n \right] \\ &\leq \bar{\mu}/n + 2\epsilon - \epsilon/n. \quad \square \end{aligned}$$

Theorem. Let $\{Z_t\}$ be the generalized linear process

$$Z_t = \sum_{j=-\infty}^{\infty} a_j e_{t-j} \quad (t = 0, \pm 1, \dots)$$

where $\sum_{j=-\infty}^{\infty} |a_j| < \infty$ and the e_t are independently and identically distributed with mean zero and finite variance $\sigma^2 > 0$. Let $\{c_t(\theta)\}$ be a sequence for which $\sup_{\theta} c_t^2(\theta) < \infty$ for each t and for which

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n |h| c_t(\theta) c_{t+|h|}(\theta) = \bar{c}(h, \theta)$$

uniformly in θ where $0 < \bar{\mu} \leq \sum_{i=-k}^k \sum_{j=-k}^k a_i a_j \bar{c}(i-j, \theta) \leq \bar{\mu} < \infty$ for all θ and all k . Then

$$S_n(\theta) = (1/\sqrt{n}) \sum_{t=1}^n c_t(\theta) Z_t$$

converges in distribution to the normal distribution with mean zero and variance

$$\tau^2(\theta) = \sigma^2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_i a_j \bar{c}(i-j, \theta) = \sum_{k=-\infty}^{\infty} \bar{c}(k, \theta) \gamma(k)$$

uniformly in θ .

Proof: The proof consists of verifying the assumptions of Lemma

1. We split $S_n(\theta)$ as

$$\begin{aligned} S_n &= (1/\sqrt{n}) \sum_{t=1}^n c_t \sum_{j=-\infty}^{\infty} a_j e_{t-j} \\ &= \sum_{t=1}^n \sum_{j=-k}^k (c_t a_j / \sqrt{n}) e_{t-j} + \sum_{t=1}^n \sum_{|j| > k} (c_t a_j / \sqrt{n}) e_{t-j} \\ &= Z_{kn} + X_{kn} \end{aligned}$$

where we have suppressed the argument θ for simplicity.

Let $\delta > 0$ be given. Since $P[|X_{kn}| > \delta] < \text{Var}(X_{kn})/\delta^2$ by Chebyshev's inequality, the first assumption of Lemma 1 may be

verified by showing $\lim_{k \rightarrow \infty} \text{Var}(X_{kn}) = 0$ uniformly in n and θ .

For given $\epsilon > 0$ there is a k_0 depending only on ϵ such that

$$(\sum_{i > k} |a_i|)^2 + (\sum_{i < -k} |a_i|)^2 < \epsilon \text{ for all } k > k_0. \text{ Then}$$

$$\frac{1}{2} \text{Var}(X_{kn}) \leq \text{Var}[\sum_t \sum_{j > k} (c_t a_j / \sqrt{n}) e_{t-j}] + \text{Var}[\sum_t \sum_{j < -k} (c_t a_j / \sqrt{n}) e_{t-j}]$$

$$= (\sigma^2/n) \sum_{i > k} \sum_{j > k} a_i a_j \sum_{t \in T} c_t c_{t-j+i}$$

$$+ (\sigma^2/n) \sum_{i < -k} \sum_{j < -k} a_i a_j \sum_{t \in T} c_t c_{t-j+i}$$

where

$$T = \{t: 1 \leq t \leq n, 1 \leq t - j + i \leq n\}$$

$$\leq (\sigma^2/n) \sum_{i > k} \sum_{j > k} |a_i a_j| (\sum_{t \in T} c_t^2)^{\frac{1}{2}} (\sum_{t \in T} c_{t-j+i}^2)^{\frac{1}{2}}$$

$$+ (\sigma^2/n) \sum_{i < -k} \sum_{j < -k} |a_i a_j| (\sum_{t \in T} c_t^2)^{\frac{1}{2}} (\sum_{t \in T} c_{t-j+i}^2)^{\frac{1}{2}}$$

$$\leq \sigma^2 (\sum_{i > k} |a_i|)^2 (n^{-1} \sum_{t=1}^n c_t^2) + \sigma^2 (\sum_{i < -k} |a_i|)^2 (n^{-1} \sum_{t=1}^n c_t^2)$$

$$\leq \sigma_c^2(0, \theta) [(\sum_{i > k} |a_i|)^2 + (\sum_{i < -k} |a_i|)^2]$$

$$< \sigma_c^2 \epsilon / a_0^2.$$

This establishes the first condition of Lemma 1.

For n larger than $2k + 1$ we may split $Z_{kn}(\theta)$ as (see Figure 1)

Figure 1.

e_{16}													a_{-3}
e_{15}												a_{-3}	a_{-2}
e_{14}											a_{-3}	a_{-2}	a_{-1}
e_{13}									a_{-3}	a_{-2}	a_{-1}	a_0	
e_{12}								a_{-3}	a_{-2}	a_{-1}	a_0	a_1	
e_{11}							a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	
e_{10}						a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3	
e_9					a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3		
e_8				a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3			
e_7			a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3				
e_6		a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3					
e_5		a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3					
e_4	a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3						
e_3	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3							
e_2	a_{-1}	a_0	a_1	a_2	a_3								
e_1	a_0	a_1	a_2	a_3									
e_0	a_1	a_2	a_3										
e_{-1}	a_2	a_3											
e_{-2}	a_3												
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}

The sum $\sum_{t=1}^{13} \sum_{j=-3}^3 c_t a_j e_{t-j}$ is obtained by multiplying each a_j by the corresponding terms on the horizontal and vertical axes and summing. The split is accomplished by segregating terms associated with those a_j between the horizontal lines.

$$\begin{aligned}
Z_{kn} &= (1/\sqrt{n}) \sum_{p=k+1}^{n-k} e_p \sum_{i=-k}^k a_i c_{p+i} \\
&+ (1/\sqrt{n}) \left(\sum_{t=1}^{2k} \sum_{j=-k}^{k-t} c_t a_j e_{t-j} + \sum_{t=n-k+1}^n \sum_{j=n-k+1-t}^k c_t a_j e_{t-j} \right) \\
&= U_{kn} + V_{kn} .
\end{aligned}$$

The variance of V_{kn} is bounded by

$$\begin{aligned}
&n^{-1} \sup_{1 \leq t \leq n} \sup_{\theta} c_t^2(\theta) \left[\mathcal{E} \left(\sum_{t=1}^{2k} \sum_{j=-k}^k |a_j| |e_{t-j}| \right)^2 \right. \\
&\quad \left. + \mathcal{E} \left(\sum_{t=n-k+1}^n \sum_{j=-k}^k |a_j| |e_{t-j}| \right)^2 \right] .
\end{aligned}$$

The term in square brackets does not vary with n so we have

$$\text{Var}(V_{kn}) \leq [n^{-1} \sup_{1 \leq t \leq n} \sup_{\theta} c_t^2(\theta)] \cdot B$$

which converges to zero as n tends to infinity uniformly in θ by Lemma 2. Consequently, if we show

$$\lim_{n \rightarrow \infty} P[U_{kn} \leq z] = N(z; 0, \sigma_k^2(\theta))$$

uniformly in θ it follows that

$$\lim_{n \rightarrow \infty} P[Z_{kn} \leq z] = N(z; 0, \sigma_k^2(\theta))$$

uniformly in θ .

Set $d_t = \sum_{i=-k}^k a_i c_{t+i}$. By Theorem 1 of Hertz (1969)

$$\sup_z |P[\sqrt{n} U_{kn}/s_n \leq z] - N(z; 0, 1)| \leq \Delta_{kn}(\theta)$$

where

$$\Delta_{kn}(\theta) = K s_n^{-3} \int_0^{s_n} \psi_n(u) du$$

$$s_n^2 = \sigma^2 \sum_{t=k+1}^{n-k} d_t^2,$$

$$\psi_n(c) = \sum_{t=k+1}^{n-k} d_t^2 \int_{|d_t e| > c} e^2 dF(e),$$

and K is a finite constant. Now

$$\begin{aligned} \Delta_{kn}(\theta) &= s_n^{-2} \int_0^1 \psi(s_n v) dv \\ &= s_n^{-2} \sum_{t=k+1}^{n-k} d_t^2 \int_0^1 \int_{|d_t e| > v s_n} e^2 dF(e) dv \\ &\leq s_n^{-2} \sum_{t=k+1}^{n-k} d_t^2 \int_0^1 \int_{e^2 > v^2 \inf_{1 \leq t \leq n} \theta (s_n^2/d_t^2)} e^2 dF(e) dv \\ &= \int_0^1 \int_{e^2 > v^2 \inf_{1 \leq t \leq n} \theta (s_n^2/d_t^2)} e^2 dF(e) dv. \end{aligned}$$

Thus, if we show that

$$\lim_{n \rightarrow \infty} \inf_{1 \leq t \leq n} \theta (s_n^2/d_t^2) = \infty$$

we will have $\lim_{n \rightarrow \infty} \Delta_{kn}(\theta) = 0$ uniformly in θ by the dominated convergence theorem and the fact that $\int_{-\infty}^{\infty} e^2 dF(e) = \sigma^2 < \infty$. Now

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{-1} s_n^2 &= \lim_{n \rightarrow \infty} \text{Var}(U_{kn}) \\ &= \sigma^2 \sum_{i=-k}^k \sum_{j=-k}^k a_i a_j \lim_{n \rightarrow \infty} n^{-1} \sum_{p=k+1}^{n-k} c_{p+i}(\theta) c_{p+j}(\theta) \\ &= \sigma^2 \sum_{i=-k}^k \sum_{j=-k}^k a_i a_j \bar{c}(i-j, \theta) \\ &= \sigma_k^2(\theta) \end{aligned}$$

uniformly in θ . Moreover, $\sigma_k^2(\theta)$ is bounded from below by \bar{l} uniformly in θ whence, for ϵ with $0 < \epsilon < \bar{u}$, there is an n_0 independent of θ , such that for $n > n_0$

$$\begin{aligned} \inf_{1 \leq t \leq n} \inf_{\theta} (s_n^2/d_t^2) &> (\bar{u}-\epsilon)/\sup_{1 \leq t \leq n} \sup_{\theta} n^{-1} d_t^2 \\ &\geq (\bar{u}-\epsilon)/\sup_{1 \leq t \leq n} \sup_{\theta} n^{-1} (\sum_{i=1}^k a_i^2) (\sum_{i=1}^k c_{t+i}^2) \\ &= (\bar{u}-\epsilon)/(2k+1) (\sum_{i=1}^k a_i^2) \sup_{1 \leq t \leq n} \sup_{\theta} n^{-1} c_t^2 \\ &\rightarrow \infty. \end{aligned}$$

Thus,

$$\begin{aligned} &|P(U_{kn} \leq z) - N(z; 0, \sigma_k^2(\theta))| \\ &\leq |P[\sqrt{n} U_{kn}/s_n \leq \sqrt{n} z/s_n] - N(\sqrt{n} z/s_n; 0, 1)| \\ &+ |N(\sqrt{n} z/s_n; 0, 1) - N(z/\sigma_k(\theta); 0, 1)| \\ &\leq \Delta_{kn}(\theta) + |N(\sqrt{n} z/s_n; 0, 1) - N(z/\sigma_k(\theta); 0, 1)| \\ &\rightarrow 0 \end{aligned}$$

uniformly in θ as n tends to infinity.

Lastly, we verify that $\lim_{k \rightarrow \infty} \sigma_k^2(\theta) = \tau^2(\theta)$ uniformly in θ .

$$\begin{aligned} |\tau^2(\theta) - \sigma_k^2(\theta)| &= |2\sigma^2 \sum_{i > |k|} \sum_{j = -\infty}^{\infty} a_i a_j \bar{c}(i-j, \theta)| \\ &\leq 2\sigma^2 \sum_{i > |k|} \sum_{j = -\infty}^{\infty} |a_i a_j \bar{c}(i-j, \theta)| \\ &\leq 2\sigma^2 \sum_{i > |k|} \sum_{j = -\infty}^{\infty} |a_i a_j| \bar{c}(0, \theta) \end{aligned}$$

because

$$\begin{aligned} n^{-1} \left| \sum_{t=1}^n -|k| \right| c_t(\theta) c_{t+|k|}(\theta) &\leq n^{-1} \sum_{t=1}^n c_t^2(\theta) \\ &\leq 2\sigma^2(\bar{u}/a_0^2) \left(\sum_{j=-\infty}^{\infty} |a_j| \right) \left(\sum_{i > |k|} |a_i| \right). \end{aligned}$$

The last term on the right does not depend on θ and may be made arbitrarily small by increasing k . \square

The result which finds application in nonlinear time series regression is the following corollary.

Corollary. Let $\{Z_t\}_{t=-\infty}^{\infty}$ be the generalized linear process

$$Z_t = \sum_{j=-\infty}^{\infty} a_j e_{t-j} \quad (t = 0, \pm 1, \dots)$$

where $\sum_{j=-\infty}^{\infty} |a_j| < \infty$ and the e_t are independently distributed with mean zero and finite variance $\sigma^2 > 0$. Let $\{c_t\}_{t=1}^{\infty}$ be a sequence of p -vectors for which the limit

$$\bar{c}(h) = \lim_{n \rightarrow \infty} \sum_{t=1}^n -|h| \left| c_t c_{t+|h|}' \right|$$

exists for all $h = 0, \pm 1, \dots$. Assume that for each non-zero p -vector λ there are finite constants \bar{l} and \bar{u} which do not depend on k such that

$$0 < \bar{l} \leq \sum_{i=-k}^k \sum_{j=-k}^k a_i a_j \lambda' \bar{c}(i-j) \lambda \leq \bar{u}.$$

Then

$$S_n = (1/\sqrt{n}) \sum_{t=1}^n c_t Z_t$$

converges in distribution to the p-variate normal with mean vector zero and variance-covariance matrix

$$\begin{aligned} V &= (\sigma^2/2) \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_i a_j [\bar{c}(i-j) + \bar{c}'(i-j)] \\ &= \frac{1}{2} \sum_{h=-\infty}^{\infty} \gamma(h) [\bar{c}(h) + \bar{c}'(h)] . \end{aligned}$$

Proof: We apply 2c.4 of Rao (1965, p. 103). Let $X \sim N_p(0, V)$. By the Theorem, $\lambda' S_n$ converges in distribution to a normal with mean zero and variance

$$\begin{aligned} \tau &= \sum_{h=-\infty}^{\infty} \gamma(h) \lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n |h| \lambda' c_t c_t' + |h| \lambda \\ &= \sum_{h=-\infty}^{\infty} \gamma(h) \lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n |h| \frac{1}{2} \lambda [c_t c_t' + |h| + c_t + |h| c_t'] \lambda \\ &= \sum_{h=-\infty}^{\infty} \gamma(h) \frac{1}{2} \lambda [\bar{c}(h) + \bar{c}'(h)] \lambda \\ &= \lambda' V \lambda . \end{aligned}$$

Thus $\lambda' S_n$ converges in distribution to $\lambda' X$ for every non-zero λ . (The matrix $\sum_{h=-\infty}^{\infty} \gamma(h) \bar{c}(h)$ is positive definite by assumption but it is not symmetric. This is the reason for the term $\frac{1}{2}[\bar{c}(h) + \bar{c}'(h)]$ in V .) \square

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