On the Bias in Flexible Functional Forms and an Essentially Unbiased Form: The Fourier Functional Form

by

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Abstract

An expenditure system is derived from a Fourier series expansion of the indirect utility function. This system has the property that the prediction bias averaged over the values of the independent variables may be made as small as desired by increasing the number of terms of the Fourier series expansion. Two consequences of this fact are that the theory of demand may be tested essentially independently of the choice of functional form and that the asymptotic distribution of other flexible form parameter estimates and test statistics may be obtained in terms of the parameters of the Fourier expenditure system. Some aspects of the bias in Translog expenditure systems are examined using these results.

1. Introduction

Much recent work on the specification of empirical expenditure systems has focused on an attempt to find an (indirect) utility function whose derived expenditure system will adequately approximate systems resulting from a broad class of utility functions. More precisely, one seeks an (indirect) utility function which yields an expenditure system whose parameters may be adjusted so that the discrepancy between the true expenditure and the approximating expenditure system is small relative to the noise in the data. Examples of this approach are in Diewert (1974), Christensen, Jorgenson and Lau (1975), and Simmons and Weiserbs (1979).

There are two methods for approximating a function that are used frequently in applications. These are Taylor's series approximations and the general class of Fourier series approximations. As examples of the latter, there is the familiar sine/cosine expansion and the possibly less familiar Jacobi, Laguerre, and Hermite expansions. The work in flexible functional forms appearing to date has used a Taylor's expansion as the approximating mechanism.

Taylor's theorem only applies locally. It applies on a neighborhood of unspecified size containing a specified value of the argument of the function being approximated - the commodity vector of a direct utility function or income normalized prices of an indirect utility function. The local applicability of the approximation suffices to translate propositions from the theory of demand into restrictions on the parameters of the approximating expenditure system; see especially Christensen, Jorgenson, and Lau (1975) and Simmons and Weiserbs (1979) in this connection. However, Taylor's theorem fails rather miserably as a means of understanding the statistical behavior of parameter estimates and test statistics; see especially Section IV of Simmons and Weiserbs (1979).

The reason for this failure is that statistical regression methods essentially expand the true function in a (general) Fourier series - not in a Taylor's series. As the sample size tends to infinity, a regression estimator $\hat{\theta}$ of the typical sort converges to that parameter value θ^* which minimizes a measure of average distance $\mathcal{B}(\theta)$ of the form

$$\mathcal{B}(\theta) = \int_{\mathcal{X}} \rho[f^*(x), f(x,\theta)]\, w(x)\, dx$$

where $\rho(y,\hat{y})$ is a measure of the distance between the true and predicted values of dependent variable determined by the estimation procedure, $\mathcal{X}$ is a set containing all possible values of the independent variable, and $w(x)$ is a density function defined on $\mathcal{X}$ giving the relative frequency with which values of the independent variable occur as sample size tends to infinity (Souza and Gallant, 1979). This is precisely the defining property of a (general) Fourier approximation of $f(x)$ by $f(x,\theta)$. A Fourier approximation attempts to minimize the average prediction bias $\mathcal{B}(\theta)$.

Due to this fact, Fourier series methods permit a natural transition from demand theory to statistical theory. The classical multivariate sine/cosine expansion of the indirect utility function leads directly to an expenditure system with the property that the average prediction bias may be made arbitrarily small by increasing the number of terms in the expansion. The key fact which permits this transition is that the classical Fourier sine/cosine series expansion approximates not only the indirect utility function to within arbitrary accuracy in terms of the $\mathcal{L}_2$ norm but also its first derivatives. Interestingly, a restriction that the Hessian of the approximating indirect utility function be positive definite is easy to impose on the Fourier flexible form; positive definiteness is not easy to impose on most other flexible forms.

The Fourier expenditure system is used as a vehicle to study potential biases resulting from the use of the Translog expenditure system. The Translog test of the theory of demand based on the equality and symmetry of coefficients as reported in Christensen, Jorgenson, and Lau (1975) is repeated using the Fourier expenditure system. Their result is confirmed. The asymptotic power curve of the Translog test of additivity is derived in terms of Fourier parameters. Parameter settings compatible with the data of Christensen, Jorgenson, and Lau are used to obtain tabular values for the power curve of the Translog additivity test. Substantial bias is found. The power curve exceeds the nominal significance level of the test when the null hypothesis is true and is relatively flat with respect to departures from the null case.

2. The Fourier Flexible Form: An Expenditure System with Arbitrarily Small Average Prediction Bias

In this section a Fourier series expansion of the indirect utility function is used to derive an expenditure system with a feature which distinguishes it from other flexible form expenditure systems. When estimated, it will approximate the true expenditure system to within an average prediction bias which may be made arbitrarily small by increasing the number of terms in the Fourier expansion.

Let q denote an N-dimensional vector of commodities, let $x = p/Y$ be the vector of normalized prices, and let $g^*(x)$ denote the consumer's true indirect utility function. The consumers utility is maximized when expenditures are allocated according to the expenditure system

$$p_i q_i / Y = [\Sigma_{i=1}^{N} x_i (\partial/\partial x_i) g^*(x)]^{-1} x_i (\partial/\partial x_i) g^*(x) \quad i = 1,2,\ldots,N$$

provided certain regularity conditions are satisfied (Diewert, 1974). No formal use is made here of these regularity conditions but it is required that the formula for the expenditure system make sense. Therefore, it is assumed that $g^*(x)$ has continuous partial derivatives and that

$$(\partial/\partial x_i) g^*(x_i) < 0$$

for all $x \in \bar{\mathcal{X}}$ where $\mathcal{X}$ is the region of approximation; the overbar denotes closure of a set.

The region of approximation is an open cube $\mathcal{X}$ constructed as follows. Let (Y_ℓ, Y_u) with $Y_\ell > 0$ be the interval of incomes over which an approximation is desired and let $(p_{\ell i}, p_{ui})$ with $p_{\ell i} > 0$ be the price intervals. Having made these choices, rescale the units of the commodities and the prices per unit such that the rescaled prices satisfy

$$0 < p_{\ell i}/Y_u < p_{ui}/Y_\ell < 2\pi \quad .$$

The region of approximation is, then,

$$\mathcal{X} = \times_{i=1}^{N}(p_{\ell i}/Y_u\,,\ p_{ui}/Y_\ell) \quad .$$

Let $\lambda = (\lambda_1,\lambda_2, \ldots, \lambda_N)'$ and $k = (k_1,k_2, \ldots, k_N)'$ be multi-indexes, i.e. N-vectors with integral components, the components of λ being non-negative. Define

$$|\lambda|^* = \Sigma_{i=1}^{N}\lambda_i \ , \qquad |k|^* = \Sigma_{i=1}^{N}|k_i| \ .$$

Partial differentiation is denoted by

$$D^\lambda f = \frac{\partial^{|\lambda|^*}}{\partial x_1^{\lambda_1}\partial x_2^{\lambda_2} \ldots \partial x_N^{\lambda_N}} f(x) \quad .$$

Differentiation is taken in the generalized sense (Rudin, 1973, §6.13) in the literature cited in this section. However, there is no need for such complexity here. If f has continuous partial derivatives of all orders up to $|\lambda|^*$ in the classical sense then the classical notion of differentiation and the generalized notion are essentially coincident. The classical notion is therefore imposed on the symbol $D^\lambda f$ here.

Let $W^{m,p}(\mathcal{X})$ denote the collection of all complex valued functions f with $|D^\lambda f|^p$ integrable over $\mathcal{X}$ for all λ with $|\lambda|^* \leq m$, a Sobolov space. Let

$$\|f\|_{m,p,\mathcal{X}} = (\Sigma_{|\lambda|^* \leq m}\int_{\mathcal{X}} |D^\lambda f|^p dx)^{1/p} \quad ,$$

the Sobolov norm. The result which motivates the Fourier expenditure system follows directly from Corollary 1 of Edmonds and Moscatelli (1977).

<u>Theorem 1</u>. Let $m \geq 2$, and for each multi-index k set

$$\theta_k(x) = e^{ik'x}$$

where i denotes the imaginary unit. If $f \in W^{m,p}(\mathcal{X})$ then there is a sequence of coefficients a_k such that

$$\lim_{K\to\infty} \| f - \Sigma_{|k|^* \le K} a_k \theta_k \|_{m-1,p,\mathcal{X}} = 0 .$$

Observe that when f is real valued the restriction $a_k = \bar{a}_{-k}$ will cause $\Sigma_{|k^*| \le K} a_k \theta_k(x)$ to be real valued without affecting the validity of Theorem 1; the overbar denotes the complex conjugate of a_k.

A more convenient form for empirical work results when the sum

$$\Sigma_{|k|^* \le K} a_k e^{ik'x}$$

is reexpressed as a double sum

$$\Sigma_{\alpha=1}^{A} \Sigma_{j=-J}^{J} a_{j\alpha} e^{ijk_\alpha' x} .$$

The requisite sequence of multi-indexes

$$\{k_\alpha : \alpha = 1, 2, \ldots, A\}$$

may be constructed from the set

$$\mathcal{K} = \{k: |k|^* \le K\}$$

as follows. First, delete from $\mathcal{K}$ the zero vector and any k whose first non-zero element is negative; i.e. (0,-1,1) would be deleted but (0,1,-1) would remain. Second, delete any k whose components have a common integral divisor; i.e. (0,2,4) would be deleted but (0,2,3) would remain. Third, arrange the k which remain into a sequence

$$\{k_\alpha : \alpha = 1, 2, \ldots, A\}$$

such that $|k_\alpha|^*$ is non-decreasing in α and such that $k_1, k_2, \ldots, k_N$ are the elementary vectors.

Assume that the observed expenditures and normalized prices are generated according to the stochastic specification

$$y_t = f^*(x_t) + e_t \qquad t = 1,2, \ldots, n$$

where

$$f^*(x) = [\Sigma_{i=1}^{N} x_i (\partial/\partial x_i)\, g^*(x)]^{-1} \begin{pmatrix} x_1\ (\partial/\partial x_1) g^*(x) \\ x_2\ (\partial/\partial x_2) g^*(x) \\ \vdots \\ x_{N-1} (\partial/\partial x_{N-1}) g^*(x) \end{pmatrix}$$

and

$$y = \begin{pmatrix} p_1 q_1 / Y \\ p_2 q_2 / Y \\ \vdots \\ p_{N-1} q_{N-1} / Y \end{pmatrix} .$$

Note that y and f^* are $(N-1)$-dimensional; the expenditures on the $N^{\underline{th}}$ commodity are obtained from $1 - \Sigma_{i=1}^{N-1} y_i$ for the observed expenditure and from $1 - \Sigma_{i=1}^{N-1} f_i^*(x)$ for the predicted expenditure. Let the errors e_t be independent and identically distributed each with zero mean vector and variance-covariance matrix Σ. Assume that, as n becomes large, the empirical distribution function of the normalized prices x_t converges to the uniform distribution over the cube $\mathfrak{X}$.

Consider as an approximation of g^* the Fourier indirect utility function

$$g(x) = a_o + b'x + \tfrac{1}{2} x' \mathbf{C} x + \Sigma_{\alpha=1}^{A} \Sigma_{j=-J}^{J} a_{j\alpha} e^{ijk_\alpha' x}$$

where

$$a_{j\alpha} = \bar{a}_{-j\alpha}$$

$$\mathbf{C} = -\Sigma_{\alpha=1}^{A} a_{o\alpha} k_\alpha k_\alpha'$$

and a_o, $a_{o\alpha}$, and b are real valued.[1] The derivatives of $g(x)$ are

$$(\partial/\partial x) g(x) = b + \mathbf{C}x + i\Sigma_{\alpha=1}^{A} \Sigma_{j=-J}^{J}\ j\ a_{j\alpha} e^{ijk_\alpha' x} k_\alpha$$

$$(\partial^2/\partial x\partial x')g(x) = -\Sigma_{\alpha=1}^{A}(a_{o\alpha} + \Sigma_{j=-J}^{J} j^2 a_{j\alpha} e^{ijk_\alpha' x})k_\alpha k_\alpha' \quad .$$

The Fourier expenditure system is

$$f_i(x,\theta) = \frac{x_i b_i + \Sigma_{\alpha=1}^{A}[-a_{o\alpha}x'k_\alpha + i\Sigma_{j=-J}^{J} j\ a_{j\alpha}e^{ijk_\alpha' x}]k_{i\alpha}x_i}{b'x + \Sigma_{\alpha=1}^{A}[-a_{o\alpha}x'k_\alpha + i\Sigma_{j=-J}^{J} j\ a_{j\alpha}e^{ijk_\alpha' x}]k_\alpha' x} \ , \ i=1,2,\ldots,N-1 \quad .$$

The system is homogeneous of degree zero in its parameters and is therefore not identified without normalization; setting $b_N = 1$ is a convenient normalization rule. Let

$$a_\alpha = (a_{o\alpha}, a_{1\alpha}, \ldots, a_{J\alpha})' \qquad \alpha = 1,2, \ldots, A \ .$$

The parameters of the system are

$$\theta = (b_1, b_2, \ldots, b_{N-1}, \alpha_1', \alpha_2', \ldots, \alpha_A')' \quad ,$$

a vector of length $N-1 + A(1+J)$. There are $N-1 + A(1+2J)$ free parameters in the vector since the complex parameters have both a real and an imaginary part.

Let $\hat{S}$ be a random matrix of order $(N-1) \times (N-1)$ with $\sqrt{n}(\hat{S} - S^*)$ bounded in probability for some positive definite matrix S^*. The nonlinear seemingly unrelated regressions estimator of θ (Gallant, 1975) is $\hat{\theta}$ which maximizes

$$s_n(\theta) = (1/n)\Sigma_{t=1}^{n} s(y_t, x_t, \hat{S}, \theta)$$

where

$$s(y,x,S,\theta) = -\tfrac{1}{2}[y - f(x,\theta)]'S^{-1}[y - f(x,\theta)] \ .$$

Subject to regularity conditions stated in Souza and Gallant (1979), $\hat{\theta}$ converges almost surely to that value θ^* which minimizes the average prediction bias

$$\mathcal{B}(\theta) = \int_{\mathcal{X}} [f^*(x) - f(x,\theta)]'(S^*)^{-1}[f^*(x) - f(x,\theta)]\ dx \ .$$

Theorem 1 and Theorem 2, below, taken together imply that the average prediction bias $\mathcal{B}(\theta^*)$ of the nonlinear seemingly unrelated regressions estimator of the Fourier expenditure system may be made as small as desired by taking A and J sufficiently large.

<u>Theorem 2</u>. Let $g^* \in W^{2,2}(\mathfrak{X})$ with continuous $(\partial/\partial x_i)g^*(x) < 0$ for all $x \in \bar{\mathfrak{X}}$ and let $f^*(x)$ be the corresponding expenditure system. Let $g_K(x,\theta_1,\theta_2,\ldots,\theta_K)$ be a sequence of functions with continuous partial derivatives in x and let $f_K(x,\theta_1,\theta_2,\ldots,\theta_K)$ be the corresponding expenditure system. Let the triangular array

$$\begin{array}{l} \theta^*_{1,1} \\ \theta^*_{1,2}\ ,\ \theta^*_{2,2} \\ \theta^*_{1,3}\ ,\ \theta^*_{2,3}\ ,\ \theta^*_{3,3} \\ \vdots \end{array}$$

minimize

$$\mathcal{B}_K(\theta_1,\theta_2,\ldots,\theta_K) = \int_{\mathfrak{X}} (f^*- f_K)'(S^*)^{-1}(f^*-f_K)\ dx$$

for $K = 1,2,\ldots$; note $\theta^*_{i,K}$ need not equal $\theta^*_{i,K+1}$. If there exists a triangular array $\{\bar{\theta}_{i,K} : i=1,2,\ldots,K\ ;\ K=1,2,\ldots\}$ such that $\bar{g}_K(x) = g_K(x,\bar{\theta}_{1,K},\bar{\theta}_{2,K},\ldots,\bar{\theta}_{K,K})$ satisfies $\bar{g}_K \in W^{2,2}(\mathfrak{X})$ and

$$\lim_{K\to\infty}\|g^*- \bar{g}_K\|_{1,2,\mathfrak{X}} = 0$$

then

$$\lim_{K\to\infty}\mathcal{B}(\theta^*_{1,K},\ \theta^*_{2,K},\ \ldots,\ \theta^*_{K,K}) = 0\ .$$

<u>Proof</u>. First it is shown that

$$\lim_{K\to\infty}\mathcal{B}(\bar{\theta}_{1,K},\ \bar{\theta}_{2,K},\ \ldots,\bar{\theta}_{K,K}) = 0\ .$$

By hypothesis, $(\partial/\partial x_i)g^*(x) \le \delta < 0$ for all $x \in \bar{\mathfrak{X}}$. Continuity on $\bar{\mathfrak{X}}$, $\lim\|g^*-\bar{g}_K\|_{1,2,\mathfrak{X}} = 0$, and

$$(\int_{\mathcal{X}} |(\partial/\partial x_i)g^* - (\partial/\partial x_i)\bar{g}_K|^2 dx)^{\frac{1}{2}} \le \|g - \bar{g}_K\|_{1,2,\mathcal{X}}$$

imply $(\partial/\partial x_i)\bar{g}_K(x) \le \eta < 0$ for all $x \in \bar{\mathcal{X}}$ and for all K sufficiently large. Then for large K, $|f_i^*(x) - \bar{f}_{i,K}(x)| \le 2$ for all $x \in \bar{\mathcal{X}}$. Further,

$$\begin{aligned} &\ell im \int_{\mathcal{X}} |x_i(\partial/\partial x_i)g^*(x) - x_i(\partial/\partial x_i)\bar{g}_K(x)|\, dx \\ &\le \ell im\, (\int_{\mathcal{X}} |x_i|^2 dx)^{\frac{1}{2}}(\int_{\mathcal{X}} |(\partial/\partial x_i)g^* - (\partial/\partial x_i)\bar{g}_K|^2\, dx)^{\frac{1}{2}} \\ &\le \ell im\, (2\pi)\, \|g^* - \bar{g}_K\|_{1,2,\mathcal{X}} \\ &= 0 \ . \end{aligned}$$

Now $(\mathcal{X},dx)$ is a finite measure space so that convergence in $\mathcal{L}_1(\mathcal{X},dx)$ implies convergence in measure. Thus, $x_i(\partial/\partial x_i)\bar{g}_K(x)$ converges in measure to $x_i(\partial/\partial x_i)g^*(x)$ as $K\to\infty$ for $i = 1, 2, \ldots, N$. It follows immediately that the expenditure shares $\bar{f}_{iK}(x)$ converge in measure to $f_i^*(x)$. Since $|f_i^* - \bar{f}_{iK}|^2$ is dominated by 4 , the dominated convergence theorem for convergence in measure implies

$$\ell im\, \Sigma_{i-1}^{N-1} \int_{\mathcal{X}} |f_i^* - \bar{f}_{iK}|^2 dx = 0 \ .$$

Let μ be the largest eigenvalue of $(S^*)^{-1}$. Then

$$0 \le \ell im\, \mathcal{B}_K(\bar{\theta}_{1,K},\ldots,\bar{\theta}_{K,K}) \le \ell im\, \mu\, \Sigma_{i=1}^{N-1} \int_{\mathcal{X}} |f_i^* - \bar{f}_{iK}|^2 dx = 0 \ .$$

The theorem follows from the fact that $(\theta_{1,K}^*, \ldots, \theta_{K,K}^*)$ minimizes $\mathcal{B}_K(\theta_1,\ldots,\theta_K)$ whence

$$0 \le \ell im\, \mathcal{B}_K(\theta_{1,K}^*, \ldots, \theta_{K,K}^*) \le \ell im\, \mathcal{B}_K(\bar{\theta}_{1,K}, \ldots, \bar{\theta}_{K,K}) = 0 \ . \quad \square$$

3. A Test of the Theory of Demand

There have been many studies that have tested the theory of demand statistically. A consice account of these studies is found in the introduction of Christensen, Jorgenson, and Lau (1975). Setting aside the well known problems with the use of aggregate data for such tests, there remains the problem of bias induced by the choice of a functional form for the expenditure system. Rejection of the null hypothesis implies rejection of either the choice of a functional form or rejection of the theory of demand or both. The implication of a significant test statistic is unclear; rejection of the theory of demand is not necessarily implied.

Tests based on the Fourier expenditure system permit clearer implications by virtue of the foregoing. Following along the same lines as Christensen, Jorgensen, and Lau (1975), a test of the theory of demand may be constructed as follows. Let

$$f(x,\theta_1,\theta_2,\ldots,\theta_{N-1}) = \begin{pmatrix} f_1(x,\theta_1) \\ f_2(x,\theta_2) \\ \vdots \\ f_{N-1}(x,\theta_{N-1}) \end{pmatrix}$$

where $f_i(x,\theta)$ is the Fourier expenditure share of the $i^{\underline{th}}$ commodity as defined in the preceding section. Note that if

$$\theta = \theta_1 = \theta_2 = \cdots = \theta_{N-1}$$

then

$$f(x,\theta_1,\theta_2,\ldots,\theta_{N-1}) = f(x,\theta) \quad .$$

Following previous usage, the restriction

$$\theta_1 = \theta_2 = \cdots = \theta_{N-1}$$

is termed the hypothesis of equality and symmetry here.

A test statistic for the hypothesis

$$\theta_1 = \theta_2 = \cdots = \theta_{N-1}$$

may be constructed from the seemingly unrelated estimator. Let $\hat{S}$ be the random matrix of the preceding section. The unconstrained estimator is $(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{N-1})$ which maximizes

$$s_n(\theta_1, \theta_2, \ldots, \theta_{N-1}) = (1/n)\Sigma_{t=1}^{n} s(y_t, x_t, \hat{S}, \theta_1, \theta_2, \ldots, \theta_{N-1})$$

where

$$s(y, x, S, \theta_1, \theta_2, \ldots, \theta_{N-1})$$
$$= -\tfrac{1}{2}[y - f(x, \theta_1, \theta_2, \ldots, \theta_{N-1})]' S^{-1} [y - f(x, \theta_1, \theta_2, \ldots, \theta_{N-1})] \,.$$

The constrained estimator is $\hat{\theta}$ which maximizes $s_n(\theta)$ as defined in the preceding section. The test statistic for equality and symmetry is

$$L = -2n[s_n(\hat{\theta}) - s_n(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{N-1})] \,.$$

One rejects the null hypothesis when L exceeds the upper α x 100 percentage point of a chi-square random variable with $(N-2)(N-1 + A(1 + 2J))$ degrees of freedom.

The Fourier expenditure system was fitted to the data of Christensen, Jorgenson, and Lau (1975). These data were obtained from Tibibian (1980) and are given in the Appendix. The multi-indices employed were

$$k_\alpha = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} ;$$

$A = 7$ and $J = 1$. These choices result in an estimate of Σ which is one half the magnitude of Σ estimated from Translog nonlinear least squares residuals; equality and symmetry constraints having been imposed in both cases. Fourier scaling as discussed in the previous section is used to estimate Σ with the Fourier expenditure system; with the Translog, prices are scaled so that each series $x_i = p_{i/Y}$ has a mean of one. There is a singularity with these data which is accommodated by fixing a_{07} at zero throughout the computations; the degrees of freedom of the test statistic are 22 .

The computed value of the test statistic for equality and symmetry is

$$L = -2(44)(-.89053 + .12783) = 67.117$$

which is significant at a level of 1% . After a correction for serial correlation the statistic is

$$L = -2(44)(-.91160 + .18207) = 64.198$$

which is significant at a level of 1% .

One concludes that the rejection of the theory of demand reported in Christensen, Jorgenson and Lau cannot be shown to result from a bias in favor of rejection induced by a choice of the Translog functional form. One is not permitted to conclude that the Translog expenditure system is free of bias from these tests, only that a bias has not been demonstrated in this instance with these data. In fact, a test of the theory of demand against an unspecified alternative is not a convenient setting in which to deal with the question of bias. The number of parameters is large, computations are therefore extremely costly, and there is no convincing means to parameterize the alternative. The Translog test for an additive indirect utility function is a much more tractable setting for an examination of bias. In the next section, a substantial bias is discovered.

4. The Power Curve of the Translog Additivity Test

If the true indirect utility function is additive then additivity may be imposed on the Fourier flexible form without affecting the ability of the Fourier expenditure system to approximate the true expenditure system. This fact allows the determination of an analytic expression for the power curve of the Translog test of additivity in terms of the parameter θ^* of the Fourier expenditure system. This power curve turns out to be shallow and biased in favor of rejection. The details follow.

Suppose that the indirect utility function is additive,

$$g^*(x) = F[\Sigma_{\alpha=1}^{N} g_\alpha^*(x_\alpha)] \quad .$$

The same expenditure system will result regardless of the choice of strictly increasing function F so it is impossible to distinguish between additivity and explicit additivity

$$g^*(x) = \Sigma_{\alpha=1}^{N} g_\alpha^*(x_\alpha)$$

from expenditure data. Therefore, only the stronger hypothesis of explicit additivity is considered here. (The same is, of course, true of homotheticity and homogeneity ; the same expenditure system results in either case.)

An explicitly additive form of the Fourier indirect utility function results when A is set to A = N; recall that the first N multi-indixes k_α are the elementary vectors. With A = N the Fourier indirect utility function may be rewritten as

$$\begin{aligned} g(x) &= \Sigma_{\alpha=1}^{N}\{a_\alpha + b_\alpha x_\alpha - \tfrac{1}{2}a_{0\alpha}x_\alpha^2 + \Sigma_{j=-J}^{J} a_{j\alpha}e^{ijx_\alpha}\} \\ &= \Sigma_{\alpha=1}^{N} g_\alpha(x) \quad . \end{aligned}$$

Theorem 1 may be applied successively to conclude that there are coefficients such that

$$\ell im_{J\to\infty}\|g_\alpha^* - g_\alpha\|_{1,2,\chi} = 0 , \qquad \alpha = 1,2,\ldots,N .$$

By the triangle inequality,

$$\ell im_{J\to\infty}\|g^* - g\|_{1,2,\chi} \leq \ell im_{J\to\infty}\Sigma_{\alpha=1}^N\|g_\alpha^* - g_\alpha\| = 0 .$$

Thus, the hypotheses of Theorem 2 are satisfied and the Fourier system is seen to retain the ability to approximate the true utility system with arbitrarily small average prediction bias.

The Translog indirect utility function yields expenditure shares

$$f_i(x,\lambda) = \frac{\alpha_i + \Sigma_{j=1}^N \beta_{ij} \ell n\, x_j}{-1 + \Sigma_{j=1}^N \beta_{Mj} \ell n\, x_j} , \qquad i = 1,2,\ldots, N-1$$

There are $N-1 + N(N+1)/2$ free parameters

$$\lambda = (a_1, a_2, \ldots, a_{N-1}, \beta_{11}, \beta_{12}, \beta_{22}, \beta_{13}, \beta_{23}, \beta_{33}, \ldots, \beta_{1N}, \beta_{2N}, \ldots, \beta_{NN})' .$$

The dependent parameters are

$$\alpha_N = -1 - \Sigma_{j=1}^{N-1} a_j ,$$

$$\beta_{ji} = \beta_{ij} \quad \text{for } i < j ,$$

$$\beta_{Mj} = \Sigma_{i=1}^N \beta_{ij} .$$

The hypothesis of explicit additivity for the Translog expenditure system takes the form $\beta_{ij} = 0$ for $i \neq j$. This hypotheses may be represented as

$$h(\lambda) = H\lambda = 0$$

where H is of order $[N(N-1)/2] \times [N-1 + N(N+1)/2]$ and is obtained from the identity of order $N-1 + N(N+1)/2$ by deleting the $N-1+N$ rows corresponding to $a_1, \ldots, a_{N-1}$ and $\beta_{11}, \ldots, \beta_{NN}$ of λ.

As before, let $\sqrt{n}(\hat{S} - S^*)$ be bounded in probability. The nonlinear seemingly unrelated regressions estimator of λ is $\hat{\lambda}$ which maximizes

$$s_n(\lambda) = (1/n)\Sigma_{t=1}^{n} s(y_t, x_t, \hat{S}, \lambda)$$

where

$$s(y,x,S,\lambda) = -\tfrac{1}{2}[y - f(x,\lambda)]'S^{-1}[y - f(x,\lambda)] \ .$$

Then, as for $\hat{\theta}$, $\hat{\lambda}$ converges almost surely to that value λ° which minimizes

$$\mathcal{B}(\lambda) = \int_{\mathcal{X}} [f^{*}(x) - f(x,\lambda)]'(S^{*})^{-1}[f^{*}(x) - f(x,\lambda)]\, dx \ .$$

To approximate $\mathcal{B}(\lambda)$, one may use $\mathcal{B}(\lambda,\theta^{*})$ where

$$\mathcal{B}(\lambda,\theta) = \int_{\mathcal{X}} [f(x,\theta) - f(x,\lambda)]'(S^{*})^{-1}[f(x,\theta) - f(x,\lambda)]\, dx \ .$$

The argument runs as follows. Restrict attention to those values of λ that yield reasonable expenditure shares $f_i(x,\lambda)$ over $\mathcal{X}$; say

$$\Lambda = \{\lambda : -\epsilon < f_i(x,\lambda) < 1 + \epsilon \ , \ x \in \bar{\mathcal{X}} \ , \ i = 1,2,\ldots,N\}$$

for some fixed $\epsilon > 0$. Note that

$$|\mathcal{B}(\lambda) - \mathcal{B}(\lambda,\theta^{*})| \le \mathcal{B}(\theta^{*}) + 2\, \mathcal{B}^{\frac{1}{2}}(\theta^{*})\, \mathcal{B}^{\frac{1}{2}}(\lambda)$$

and $\mathcal{B}(\lambda)$ is bounded over Λ by $\mu(N-1)(2+\epsilon)^2$ where μ is the largest eigenvalue of $(S^{*})^{-1}$. As seen in the previous section, $\mathcal{B}(\theta^{*})$ may be made arbitrarily small by taking A and J sufficiently large independently of the value of $\lambda \in \Lambda$. Thus, λ° can be computed as that value of λ which minimizes $\mathcal{B}(\lambda,\theta^{*})$ and the error of approximation may be made arbitrarily small by taking A and J sufficiently large.

The Wald test and the Lagrange multiplier test for the hypothesis

$$h(\lambda^{\circ}) = 0$$

are distributed asymptotically as non-central chi squared random variables each with $N(N-1)/2$ degrees of freedom (Souza and Gallant, 1979), the non-centrality parameter is

$$\alpha^{\circ} = n\, \lambda^{\circ\prime} H'(HV^{\circ}H')^{-1} H\lambda^{\circ}/2$$

where

$$V^{\circ} = (\mathcal{J}^{\circ})^{-1}\mathcal{I}^{\circ}(\mathcal{J}^{\circ})^{-1}$$

$$\mathcal{I}^{\circ} = (1/n)\Sigma_{t=1}^{n}[(\partial/\partial\lambda')f(x_t,\lambda^{\circ})]'(S^{*})^{-1}[\Sigma + \delta(x_t,\lambda^{\circ},\theta^{*})\delta'(x_t,\lambda^{\circ},\theta^{*})](S^{*})^{-1} [(\partial/\partial\lambda')f(x_t,\lambda^{\circ})]$$

$$\mathcal{J}^{\circ} = (1/n)\Sigma_{t=1}^{n}[(\partial/\partial\lambda')f(x_t,\lambda^{\circ})]'(S^{*})^{-1}[(\partial/\partial\lambda')f(x_t,\lambda^{\circ})]$$

$$- (1/n)\Sigma_{t=1}^{n} \Sigma_{i=1}^{N-1} \Sigma_{j=1}^{N-1} \delta_i(x_t,\lambda^{\circ},\theta^{*})\, S^{*ij}(\partial^2/\partial\lambda\partial\lambda')f_j(x_t,\lambda^{\circ})$$

$$\delta(x_t,\lambda^{\circ},\theta^{*}) = f(x_t,\theta^{*}) - f(x_t,\lambda^{\circ})$$

and S^{*ij} denotes the elements of S^{*-1}. The asymptotic non-null distribution of the analog of the likelihood ratio test is also given in Souza and Gallant (1979) but it does not have a tabled null distribution in this case. Thus, it is of no practical importance. The arguments supporting the substitution of $f(x,\theta^{*})$ for $f^{*}(x)$ in these formulas are similar to those supporting the use of $\mathcal{B}(\lambda,\theta^{*})$ for $\mathcal{B}(\lambda)$.

The choice of S^{*} for use in these formulas presents somewhat of a problem. The simplest choice is to take $S^{*} = \Sigma$ which is equivalent to assuming that either Σ is known or that it may be estimated with negligible bias. It is, of course, always possible to obtain Σ with negligible bias, one need only fit a polynomial in x of suitably high degree to each expenditure share y_i and compute $\hat{S}$ from the residuals (Gallant, 1979). The alternative approach is to assume that $\hat{S}$ was computed from translog residuals and account for the resulting bias. For example, one might compute $\hat{S}$ from nonlinear least squares residuals subject to the equality and symmetry across equation constraint. This is equivalent to taking $\hat{S} = I$ in the nonlinear seemingly unrelated regressions method whence

$$S^* = \Sigma + \int_{\mathcal{X}} \delta(x,\lambda^{\circ\circ},\theta^*)\, \delta'(x,\lambda^{\circ\circ},\theta^*)\, dx$$

where $\lambda^{\circ\circ}$ minimizes

$$\int_{\mathcal{X}} \delta'(x,\lambda,\theta^*)\, \delta(x,\lambda,\theta^*)\, dx \quad .$$

Another possibility is to compute $\hat{S}$ from unconstrained translog residuals. In view of the variety of choices available for S^* and the additional complexity entailed, it seems that the simplest choice $S^* = \Sigma$ contributes more to understanding. From the data of the Appendix, a variance-covariance matrix Σ was computed from Fourier expenditure system residuals with equality and symmetry imposed on the fit; $A = 7$ and $J = 1$. This variance-covariance matrix was rescaled upward by a factor of two.

A smooth transition between the extremes of additivity and its absence was obtained as follows. The parameter θ^* was computed by fitting the Fourier expenditure system to the data of the Appendix by nonlinear seemingly unrelated regressions with this choice of Σ , with equality and symmetry imposed, and with the constraint

$$\left(\Sigma_{\alpha=4}^{7}\ \Sigma_{j=-1}^{1}\ |a_{j\alpha}|^2\right)^{\frac{1}{2}} = K$$

imposed. The choice $K = 0$ yields the null case. The remaining lines of Table 1 correspond to increasingly larger values of K and the last line corresponds to an unconstrained fit. These parameter choices are realistic in that they yield expenditure shares in accord with the expenditure shares in the data of the Appendix as revealed by visual inspection of plots of observed and predicted shares against time.

The Translog test of explicit additivity (with equality and symmetry as a maintained hypothesis) is seriously flawed as seen in Table 1. The actual size of the test is much larger than the nominal significance level of .010 and the power curve is relatively flat compared to the power of a test based on the Fourier expenditure system. The Translog power curve does increase locally, as one might expect, but it falls off again as departures from the null case become more extreme.

Table 1 about here

Table 1. Tests for an Additive Indirect Utility Function

	Fourier		Translog	
K	Noncentrality	Power	Noncentrality	Power
.0	.0	.010	8.9439	.872
.00046	.0011935	.010	8.9919	.874
.0021	.029616	.011	9.2014	.884
.0091	.63795	.023	10.287	.924
.033	4.6689	.260	14.268	.987
.059	7.8947	.552	15.710	.933
.084	82.875	1.000	13.875	.984
unconstrained	328.61	1.000	10.230	.922

5. Parameter Reduction

As with other flexible functional forms, the number of parameters in a Fourier expenditure system becomes unreasonably large as the number of commodities increases. This problem can be partially alleviated by imposing a convexity restriction and by deleting selected multi-indexes.

Convexity is imposed on the Fourier indirect utility function rewritten as

$$g(x) = a_o + b'x + \Sigma_{\alpha=1}^{A}[-\tfrac{1}{2}a_{o\alpha}(k_\alpha'x)^2 + \Sigma_{j=-2J}^{2J} a_{j\alpha}e^{ijk_\alpha'x}]$$

by setting

$$a_{o\alpha} = -\Sigma_{s=-J}^{J} c_{s\alpha}\bar{c}_{s\alpha} \qquad \alpha = 1,2,\ldots,A$$

$$a_{j\alpha} = (-1/j^2)\Sigma_{s=-J}^{J} c_{s\alpha}\bar{c}_{s-j,\alpha} \qquad \alpha = 1,2,\ldots,A; \quad j = 1,2,\ldots,2J$$

where the free parameters satisfy

$$c_{j\alpha} = \bar{c}_{-j\alpha} \qquad \alpha = 1,2,\ldots,A; \quad j = 0,1,2,\ldots,J$$

$$c_{o\alpha} \geq 0 \qquad \alpha = 1,2,\ldots,A;$$

$$c_{j\alpha} = 0 \qquad \alpha = 1,2,\ldots,A; \quad |j| > J \quad .$$

The restriction is sufficient for $g(x)$ to be a convex function; it is not necessary save in the case when $A \leq N$. Also, the unbiasedness property of the Fourier system is lost when the restriction is imposed. However, since the restriction can be tested in an application, the lack of necessity and the loss of unbiasedness are not serious problems. One would think that any serious distortion of the fit caused by the convexity restriction would be detected by testing the restriction. It reduces the number of parameters by roughly one half, holding the length of θ constant.

Further parameter reduction may be achieved as follows. Rewrite the

Fourier indirect utility function as

$$g(x) = a_o + b'x + \Sigma_{\alpha=1}^{A}\mu_\alpha(k_\alpha' x)$$

where

$$\mu_\alpha(z) = -\tfrac{1}{2}\, a_{o\alpha} z^2 + \Sigma_{j=-2J}^{2J}\, a_{j\alpha} e^{ijz} \quad .$$

Written thus, the Fourier indirect utility function is seen to be additive not in each of the normalized prices x_α but rather in price indexes $k_\alpha' x$. The components of the multi-indexes k_α are the weights which make up the price index. Thus, either an upward or downward testing sequence may be employed to determine optimal set of multi-indexes in a given application.

The convexity claim is verified as follows. The Hessian of $g(x)$ is

$$(\partial^2/\partial x \partial x')g(x) = \Sigma_{\alpha=1}^{A}(d^2/dz^2)\mu_\alpha(k_\alpha' x)k_\alpha k_\alpha'$$

and a sufficient condition for a positive semi-definite Hessian is that for each α

$$(d^2/dz^2)\mu_\alpha(z) \geq 0 \qquad 0 \leq z \leq 2\pi \ .$$

But, under the restrictions,

$$\begin{aligned}(d^2/dz^2)\mu_\alpha(z) &= -a_{o\alpha} - \Sigma_{j=-2J}^{2J} a_{j\alpha} j^2 e^{ijz} \\ &= \Sigma_{j=-2J}^{2J}(\Sigma_{s=-J}^{J} c_{s\alpha}\bar{c}_{s-j,\alpha})e^{ijz} \\ &= \Sigma_{s=-J}^{J}\Sigma_{p=-2J-s}^{2J-s}\, c_{s\alpha}\bar{c}_{-p\alpha} e^{ipz + isz} \\ &= \Sigma_{s=-J}^{J}\, c_{s\alpha} e^{isz}\, \Sigma_{p=-J}^{J}\, \bar{c}_{-p\alpha} e^{ipz} \\ &= (\Sigma_{s=-J}^{J} c_{s\alpha} e^{isz})\overline{(\Sigma_{p=-J}^{J} c_{-p\alpha} e^{-ipz})} \\ &= (\Sigma_{s=-J}^{J} c_{s\alpha} e^{isz})\overline{(\Sigma_{s=-J}^{J} c_{s\alpha} e^{isz})} \\ &\geq 0 \ .\end{aligned}$$

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Footnote

[1] Experience in other contexts indicates that the number of terms in a Fourier approximation of a function can be reduced considerably if a linear term $b'x$ is included. The term $x'Cx$ is included so that the expression for the Hessian has the form of a weighted sum of rank one positive semi-definite matrices with scalar Fourier series expansions of $k_\alpha'x$ as weights. This facilitates the approximation of a convex function; see Section 5.

Table 2. Data of Christensen, Jorgenson, and Lau (1975)

	DURABLES		NONDURABLES		SERVICES	
Year	Quantity	Price	Quantity	Price	Quantity	Price
1929	28.9645	33.9	98.1	38.4	96.1	31.6
1930	29.8164	32.2	93.5	36.4	89.5	32.1
1931	28.9645	31.4	93.1	31.1	84.3	30.9
1932	26.8821	23.9	85.9	26.5	77.1	28.8
1933	25.3676	31.3	82.9	26.8	76.8	26.1
1934	24.6104	27.7	88.5	30.2	76.3	26.8
1935	22.3387	28.8	93.2	31.5	79.5	26.8
1936	24.1371	32.9	103.8	31.6	83.8	27.2
1937	24.1371	29.0	107.7	32.7	86.5	28.3
1938	26.6928	28.4	109.3	31.1	83.7	29.1
1939	26.4088	30.5	115.1	30.5	86.1	29.2
1940	27.0714	29.4	119.9	30.9	88.7	29.5
1941	28.4912	28.9	127.6	33.6	91.8	30.8
1942	29.5325	31.7	129.9	39.1	95.5	32.4
1943	28.6806	38.0	134.0	43.7	100.1	34.2
1944	28.8699	37.7	139.4	46.2	102.7	36.1
1945	28.3966	39.0	150.3	47.8	106.3	37.3
1946	26.6928	44.0	158.9	52.1	116.7	38.9
1947	28.3966	65.3	154.8	58.7	120.8	41.7
1948	31.6149	60.4	155.0	62.3	124.6	44.4
1949	35.8744	50.4	157.4	60.3	126.4	46.1
1950	38.9980	59.2	161.8	60.7	132.8	47.4
1951	43.5414	60.0	165.3	65.8	137.1	49.9
1952	48.0849	64.2	171.2	66.6	140.8	52.6
1953	49.8833	57.5	175.7	66.3	145.5	55.4
1954	53.1016	68.3	177.0	66.6	150.4	57.2
1955	55.4680	63.5	185.4	66.3	157.5	58.5
1956	58.8756	62.2	191.5	67.3	164.8	60.2
1957	61.6206	56.5	194.8	69.4	170.3	62.2
1958	65.3122	66.7	196.8	71.0	175.8	64.2
1959	65.7854	63.3	205.0	71.4	184.7	66.0
1960	68.6251	73.1	208.2	72.6	192.3	68.0
1961	70.6129	72.1	211.9	73.3	200.0	69.1
1962	71.5594	72.4	218.5	73.9	208.7	70.4
1963	73.5472	72.5	223.0	74.9	217.6	71.7
1964	77.2387	76.3	233.3	75.8	229.7	72.8
1965	81.9715	82.3	244.0	77.3	240.7	74.3
1966	87.4615	84.3	255.5	80.1	251.6	76.5
1967	93.8981	81.0	259.5	81.9	264.0	78.8
1968	99.5774	81.0	270.2	85.3	275.0	82.0
1969	106.7710	94.4	276.4	89.4	287.2	86.1
1970	109.1380	85.0	282.7	93.6	297.3	90.5
1971	115.2900	88.5	287.5	96.6	306.3	95.8
1972	122.2000	100.0	299.3	100.0	322.4	100.0

Source: Tibibian (1980)