ESTIMATING SUBSTITUTION ELASTICITIES WITH THE FOURIER COST FUNCTION: SOME MONTE CARLO RESULTS

by

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Revised November 1982

Institute of Statistics Mimeograph Series No. 1614

ABSTRACT

The Fourier flexible form possesses desirable asymptotic properties not shared by other flexible forms such as the translog, generalized Leontief, and the generalized Box-Cox. Here we present a Monte Carlo study designed to assess whether or not these asymptotic properties take hold in situations commonly encountered in practice. A three-input, homothetic version of the generalized Box-Cox cost function is used to generate various technologies and the Fourier cost function is used to approximate the associated elasticities of each. We find a response curve for bias which is relatively flat in the design parameters. Since the logarithmic version of the Fourier flexible form appears to approximate the entire range adequately with a small number of parameters, we conclude that its superior asymptotic properties do carry over and that it dominates fixed-parameter flexible forms in practice.

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1. Introduction and Review of Past Work

The specification of "flexible" functional forms in consumer demand or production studies involves the use of an expression, perhaps derived with a Taylor's series, Box-Cox transformation, etc., which locally approximates some unknown function. These flexible forms such as the translog or generalized Leontief appear to perform well in approximation over certain regions of the parameter space and in some applications, not so well in others. Caves and Christensen (1980) show that these two forms do not necessarily satisfy the restrictions of monotonicity and quasi-convexity, for instance, over the entire range of prices and income in consumer demand analysis, and that the ranges of application differ for the two forms. Similarly, Wales (1977) finds cases in which each outperforms the other in approximating preferences with a constant elasticity of substitution between two goods, and that both forms can violate these restrictions. He observes that in view of this possibility, rejections of consumer theory common in this area may be due to the chosen form's violation of regularity conditions, rather than an absence of utility-maximizing behavior. Gallant (1982) argues that rejection of hypotheses may also be due in some cases to biases induced by the departure of the true function from the flexible representation chosen, even when the latter satisfies the relevant theoretical restrictions. It is this problem of specification error which we consider in this paper.

Other examples of these flexible form comparisons can be found. The results of Berndt, Darrough, and Diewert (1977) strongly favor the translog on Bayesian grounds over the generalized Leontief and generalized Cobb-Douglas, for Canadian expenditure data. A parametric approach is used in the production study of Berndt and Khaled (1979). They use the generalized Box-Cox cost function, and test for the other forms as special cases, using U.S. manufacturing data.[1] They reject the generalized square-root quadratic; the generalized Leontief cannot be rejected, while results do not seem to favor the translog. The latter cannot be tested through parametric restrictions on the generalized Box-Cox, since it is a limiting case. However, they find that the value of the likelihood function drops significantly for values of the Box-Cox parameter near the translog case of zero. Guilkey and Lovell (1980), White (1980), and Gallant (1981) also provide cases which indicate some limitations of the translog.

The results of Guilkey, Lovell, and Sickles (1981) indicate that the quality of approximation provided by a translog form deteriorates as partial elasticities of substitution depart from unity (as do Guilkey and Lovell (1980) and Wales (1977)), or as they depart from one another. However, they accept the translog as the least objectionable of flexible forms tested. The translog dominates the generalized Leontief, they find, except when all Allen-Uzawa partial elasticities of substitution are small and positive (a case in which Wales (1977) found both forms adequate). The generalized Box-Cox is not included in their study due to computational costs.

The Fourier flexible form possesses desirable asymptotic properties not shared by these other flexible forms such as globally consistent estimates of elasticities, asymptotically size α tests, and globally negligible prediction bias. These results are obtained by letting the number of parameters depend on the sample size (El Badawi, Gallant, and Souza (1982)) thus leaving open the question as to whether these desirable asymptotic properties take hold when the model has only a small number of parameters.[2] Comparisons by Kumm (1981) against the generalized Box-Cox form for a fixed data set suggest that this is the case. In this paper, we use Monte Carlo methods to examine the ability of the logarithmic version of the Fourier form, introduced by Gallant (1982), to approximate each of the technologies listed above, using only a small number of parameters.

In our analysis, we follow Guilkey, Lovell, and Sickles (1981), who suggest in their study of cost functions that if properties of functional forms are of interest, the appropriate criterion is not how well data generated by unknown technologies are modeled. They choose instead to "begin with known technology and examine the ability of various forms to track that technology" (page 2). They consider the translog, generalized Leontief, and generalized Cobb-Douglas, examining the approximation with these forms of an "almost homothetic" technology.

Our design approach is to choose nine technologies of the generalized Box-Cox form according to a central composite rotatable design as described in, for example, Cochran and Cox (1957). We orient the design so that a Leontief technology, the generalized square-root quadratic flexible form, two cases of the generalized Leontief flexible form, and the translog appear as points in the design. The two dimensions of the layout are a measure of dispersion in the substitution matrix and the parameter λ of the generalized Box-Cox form. Setting λ equal to zero generates the translog, a special case of the Fourier form, while increasingly larger settings of λ depart from this class and therefore generate technologies that should be increasingly harder for the Fourier form to track with a modest number of parameters. Then, following Hendry (1982), fitting a quadratic to the Monte Carlo results allows exploration of the entire response surface. By choosing as design points a variety of technologies we provide a more interesting test of the usefulness of any flexible form, as well as indicating how robust the approximation might be to changes in the underlying technology.

Implicit in the Guilkey, Lovell, and Sickles (1981) approach is the observation that the standard methodology for fitting demand systems generated from a cost function generates an errors-in-variables problem, even if the model is correctly specified. Using the fact that the translog is a special case of the Fourier form, we adjust the variance in the data-generating mechanism so that the errors-in-variables problem is held at an acceptable level in that technology. The Monte Carlo experiment is designed to detect specification error bias over and above the errors-in-variables bias. The results indicate that if standard methodology is viable at all then the Fourier form includes, as a practical matter, all technologies and error settings that can reasonably be anticipated a priori.

We proceed as follows. The generalized Box-Cox cost function and its special cases are described, and expressions for shares and substitution elasticities are provided. Next, the logarithmic version of the Fourier form is described. Our experimental design is then presented, along with the method by which experimental data have been generated. Finally, the results of Monte Carlo experiments are examined.

2. The Generalized Box-Cox Cost Function

The generalized Box-Cox cost function is taken from Berndt and Khaled (1979). We impose linear homogeneity in prices and homotheticity of the underlying technology, which simplifies the expression for costs to

$$C = [(2/\lambda) \sum_i \sum_j \gamma_{ij} \, P_i^{\frac{\lambda}{2}} P_j^{\frac{\lambda}{2}}]^{\frac{1}{\lambda}} h(y).$$

Since interest is in the elasticities of substitution, output y is taken to be unity. So long as interest centers on the curvature of isoquants and not on their position, this is justified. In the case of a homothetic technology, output drops out of the expression for the Allen-Uzawa partial elasticity of substitution.

Having chosen this case of the generalized Box-Cox cost function, special cases are obtained as in Berndt and Khaled (1979), by choice of the Box-Cox parameter λ. The generalized square-root quadratic obtains with $\lambda = 2$, and is of the form

$$C = [\sum_i \sum_j \gamma_{ij} P_i P_j]^{\frac{1}{2}}.$$

The homothetic generalized Leontief results from setting $\lambda = 1$, and

$$C = 2\sum_i \sum_j \gamma_{ij} P_i^{\frac{1}{2}} P_j^{\frac{1}{2}}.$$

Finally, the translog is a limiting case, when λ approaches zero:

$$\ln C = \alpha_0 + \sum_i \alpha_i \cdot \ell n\ P_i + \frac{1}{2} \sum_i \sum_j \gamma_{ij} \cdot \ell n\ P_i \cdot \ell n\ P_j.$$

Differentiation of the cost function in each case, with re-scaling by (P_i/C) gives expressions for factor shares

$$S_i = 2\ P_i^{\frac{\lambda}{2}} (\sum_j \gamma_{ij}\ P_j^{\frac{\lambda}{2}})/\lambda C^{\lambda}, \quad i=1,\ldots n.$$

Substitution elasticities are likewise obtained, as special cases of expressions given by Berndt and Khaled (1979, page 1225):

$$\sigma_{ii} = 1 - \lambda + \gamma_{ii} \frac{P_i^{\lambda}}{S_i^2} \cdot c^{-\lambda} + \frac{\lambda}{2S_i} - \frac{1}{S_i} \quad i=1,\ldots,n$$

and

$$\sigma_{ij} = 1 - \lambda + \gamma_{ij} \frac{(P_i P_j)^{\frac{\lambda}{2}}}{S_i S_j} c^{-\lambda} \quad (i \neq j).$$

The corresponding expressions for shares and substitution elasticities in the translog case are

$$S_i = \alpha_i + \sum_j \gamma_{ij} \ln P_j, \qquad i=1,\ldots,n;$$

$$\sigma_{ii} = 1 + \frac{\gamma_{ii}}{S_i^2} - \frac{1}{S_i}, \qquad i=1,\ldots,n;$$

and

$$\sigma_{ij} = 1 + \frac{\gamma_{ij}}{S_i S_j}, \qquad i \neq j=1,\ldots,n.$$

The share expressions in this section are used along with chosen parameter values in the generalized Box-Cox cost function to generate observations on factor shares, given a set of prices. These generated share and price data are then used to fit the logarithmic version of the Fourier cost function.

3. The Fourier Cost Function

Unlike other flexible forms found in the literature, the Fourier flexible form allows a variable number of parameters. Gallant (1982) has suggested letting the number of parameters depend on the sample size. For the least average bias, one should use as large a number of parameters as the data permit. In this section, the logarithmic version of the Fourier cost function is described. The discussion is brief; the reader should refer to Gallant (1981, 1982) and El Badawi, Gallant, and Souza (1982) for complete description.

The Fourier form is different from other flexible cost functions, in that the criterion used for goodness-of-fit is the Sobolev norm. Interest in production studies generally centers on a cost function, factor demands or share equations, and elasticities of substitution. Therefore, it is important to closely approximate not only the true cost function, but its first and second derivatives. Higher-order approximation errors are unimportant. The Sobolev norm is therefore the relevant measure of distance. The logarithmic version of the Fourier form serves as a function

$$g_{K_T}(x|\theta)$$

which approximates some true cost function

$$g(x)$$

as closely as is desired in Sobolev norm[3] where K_T denotes the length of the parameter vector θ and x is a vector of logged input prices. We seek close approximation of $g(x)$, $\nabla g(x)$, and $\nabla^2 g(x)$, where

$$\nabla g(x) = (\partial/\partial x)g(x)$$

and

$$\nabla^2 g(x) = (\partial^2/\partial x \partial x')g(x).$$

The standard cost function

$$C(p,y)$$

gives the minimum attainable cost of producing output y with inputs

$$q_1 = (q_1,\ldots,q_n)',$$

given the factor prices

$$p = (p_1,\ldots,p_n)'.$$

When the production technology is homothetic,

$$C(p,y) = h(y)c(p).$$

If C is the true cost function, then the firm's factor demands are

$$q = (\partial/\partial p)C(p,y),$$

by Shephard's Lemma (Diewert (1974), Varian (1978)). Differentiating once more, the elasticity of substitution

$$\sigma_{ij} = \frac{C(P,y)\,(\partial^2/\partial P_i \partial P_j)\,C(P,y)}{(\partial/\partial P_i)\,C(P,y)\,(\partial/\partial P_j)C(P,y)} = \sigma_{ij}\,(p,y)$$

and (output-constant) price elasticity

$$\nu_{ij} = \sigma_{ij} \cdot s_j = \nu_{ij}(p,y)$$

are obtained. It is these quantities and their standard errors which may be estimated with the Fourier cost function.

The restatement of the problem in logarithmic quantities provided by Gallant (1982) yields equivalent expressions. We work with logged prices and define

$$X_i = \ell n P_i + \ell n a_i, \quad i=1,\ldots,n,$$

and

$$v = \ell n\ y + \ell n\ a_{n+1}.$$

The ℓn a's are location-shifters, used to make all log prices positive,so that each price Pi is replaced by

$$e^{X_i}/a_i$$

and y by

$$e^{V}/a_{n+1}.$$ [4]

The cost function becomes

$$k(x,v) = \ln c(\frac{e^{X_1}}{a_1}, \frac{e^{X_2}}{a_2}, \ldots, \frac{e^{X_n}}{a_n}, \frac{e^{V}}{a_{n+1}})$$

where

$$x = (x_1, \ldots, x_n)' .$$

We have now that

$$(\partial/\partial p)c(p,y) = c(p,y)\ P^{-1}\nabla k,$$

and

$$(\partial^2/\partial p \partial p')c(p,y) = c(p,y)\ P^{-1}[\nabla^2 k + \nabla k \nabla k' - \mathrm{diag}(\nabla k)]\ P^{-1},$$

where

$$\nabla k = (\partial/\partial x)k(x,v),$$

$$\nabla^2 k = (\partial^2/\partial x \partial x')k(x,v),$$

and

$$P = \mathrm{diag}(p);$$

homotheticity now implies

$$k(x,v) = h(v) + g(x).$$

The vector of shares s is obtained by

$$s = (\partial/\partial x)k(x,v)$$

which, with homotheticity imposed, becomes

$$s = (\partial/\partial x)g(x).$$

The elasticities of substitution are obtained as elements of

$$\Sigma = ((\sigma_{ij})) = [\mathrm{diag}(\nabla k)]^{-1}[\nabla^2 k + \nabla k \nabla k' - \mathrm{diag}(\nabla k)]\ [\mathrm{diag}(\nabla k)]^{-1}$$

and

$$\nu = ((\nu_{ij})) = \Sigma \cdot \mathrm{diag}(\nabla k).$$

Since only the shape of an individual isoquant is of interest when the technology is homothetic, ν , Σ , and s depend only on prices. Output and total costs are unimportant.[5]

We are left, then, with share equations, the coefficients of which may be estimated statistically to recover information about the underlying parameters of interest. Approximation to the true cost function $g(\cdot)$ is desired over a rectangle in the positive orthant, of a size chosen in applications according to the preferences of the researcher. Once chosen, the rectangle's dimensions must be rescaled so that no edge is longer than 2Π, since the Fourier form is periodic in prices. We rescale so that all observed prices are contained in $[0,2\Pi]$. This is extremely important, and cannot be omitted from the estimation procedure.

We use in the expressions which follow the concept of a multi-index, which is simply an n-vector with integer components[6]

$$k = (k_1,\ldots,k_n)'$$

with norm defined as $|k|^* = \sum_{i=1}^{n} |ki|$.

A multi-index can be used to denote partial differentiation, viz.,

$$D^{\lambda} g(x) = \frac{\partial^{|\lambda|^*}}{\partial x_1^{\lambda_1}\, \partial x_2^{\lambda_2} \ldots \partial x_n^{\lambda_n}} g(x),$$

for

$$\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n)'.$$

Multi-indexes also provide a convenient means for the expression of multivariate Fourier series. We consider all elementary multi-indexes of length less than an arbitrary constant. A set of elementary multi-indexes is obtained by deleting all multi-indexes which can be expressed as an integer multiple of a multi-index with smaller norm, plus any multi-indexes in which the first non-zero entry is negative. To obtain linear homogeneity in prices, the set is further reduced to those vectors which involve contrasts among the prices. Reduction to this set of elementary multi-indexes permits more convenient representations of the Fourier form. See Gallant (1982) for details and Figure 1 for an illustration.

The following expression represents the Fourier flexible form used as an approximation to the true cost function g:

$$g_{K_T}(x|\theta) = u_0 + b'x + \frac{1}{2}x'Cx + \sum_{\alpha=1}^{A}\{u_{0\alpha} + 2\sum_{j=1}^{J}[u_{j\alpha}\cos(j\lambda k_\alpha' x)$$

$$- v_{j\alpha}\sin(j\lambda k_\alpha' x)]\}$$

with

$$C = -\lambda^2 \sum_{\alpha=1}^{A} u_{0\alpha} k_\alpha k_\alpha'.$$

The derivatives of $g_{K_T}(x|\theta)$ are

$$(\partial/\partial x) g_{K_T}(x|\theta) = b - \lambda \sum_{\alpha=1}^{A}\{u_{0\alpha}\lambda\ k_\alpha' x + 2\sum_{j=1}^{J} j[u_{j\alpha}\sin(j\lambda k_\alpha' x)$$

$$+ v_{j\alpha}\cos(j\lambda k_\alpha' x)]\}\ k_\alpha$$

$$(\partial^2/\partial x \partial x') g_{K_T}(x|\theta) = -\lambda^2 \sum_{\alpha=1}^{A}\{u_{0\alpha} + 2\sum_{j=1}^{J} j^2[u_{j\alpha}\cos(j\lambda k_\alpha' x)$$

$$- v_{j\alpha}\sin(j\lambda\ k_\alpha' x)]\}\ k_\alpha k_\alpha'.$$

λ is the scaling factor which is used to keep all prices (and output v, when it is included) in the interval $(0, 2\Pi)$. We have chosen

$$\lambda = 6/\ \max\{Xi\}.$$

The parameters of the Fourier cost function $g_{K_T}(\cdot)$ are

$$\Theta = (0, b_1, b_2, b_3, u_{0\alpha}, u_{j\alpha}, v_{j\alpha})' \quad \alpha = 1,\ldots,n;\ j = 1,\ldots,J.$$ [7]

When A = 3 and J = 1 the length of Θ is 13, and three multi-indexes are used. When A = 6, the length of θ is 22, and six are used.

It is convenient to think of $g_{K_T}(x|\theta)$ as being the sum of a translog

$$u_0 + b'x + \frac{1}{2} x'Cx,$$

and the sum of A univariate Fourier expansions in the directions determined by the vectors k_α. We impose the homogeneity restriction

$$\sum_{i=1}^{n} bi = 1.$$

The cost function associated with the $K_T = 22$ case is

$$\begin{aligned}
g_{K_T}(x|\Theta) &= u_0 + b'x + \frac{1}{2} x'Cx \\
&+ u_{01} + 2\,u_1 \cos[\lambda\ (x_2 - x_3)] - v_1 \sin[\lambda\ (x_2 - x_3)] \\
&+ u_{02} + 2\,u_2 \cos[\lambda\ (x_1 - x_2)] - v_2 \sin[\lambda\ (x_1 - x_2)] \\
&+ u_{03} + 2\,u_3 \cos[\lambda (x_1 - x_3)] - v_3 \sin[\lambda (x_1 - x_3)] \\
&+ u_{04} + 2u_4 \cos[\lambda (x_1 - 2x_2 + x_3)] - v_4 \sin[\lambda (x_1 - 2x_2 + x_3)] \\
&+ u_{05} + 2u_5 \cos[\lambda (x_1 + x_2 - 2x_3)] - v_5 \sin[\lambda (x_1 + x_2 - 2x_3)] \\
&+ u_{06} + 2u_6 \cos[\lambda (2x_1 - x_2 - x_3)] - v_6 \sin[\lambda (2x_1 - x_2 - x_3)].
\end{aligned}$$

To reduce the length of θ to 13, then, the coefficients

$$\{u_{o4}, u_4, v_4, u_{05}, u_5, v_5, u_{06}, u_6, v_6\}$$

are set equal to zero.

Figure 1: Multi-indexes for the 22-parameter

Fourier cost function

0	1	1	1	1	2
1	-1	0	-2	1	-1
-1	0	-1	1	-2	-1

4. Estimation

Technologies for each of the design points were estimated with the length of the Fourier parameter vector set at $K_T=13$ and $K_T=22$. Theorem 2 of Gallant (1982) shows that a cost function satisfying positive linear homogeniety can be approximated by the Fourier flexible form with all multi-indexes which do not involve contrasts among input prices deleted, and with

$$\sum_{i=1}^{n} b_i = 1.$$

This is accomplished by using the multi-indexes in Figure 1. The $K_T=13$ case involves the three multi-indexes of norm 2; $K_T=22$ involves also the next three, of norm 4.

The estimation procedure is discussed fully in Section 4.1 of Gallant (1982). The model fitted to share and price data is

$$y_t = (s_1, s_2)'_t = f(x_t|\theta) + e_t = Z'_t\theta + e_t$$

since the model is linear in the parameters.[8] Zellner's (1962) Seemingly-Unrelated Regressions technique is used to obtain an estimate for θ. We choose as the estimate of θ that vector $\hat{\theta}$ which minimizes

$$S(\theta,\Sigma) = \frac{1}{T}\sum_{t=1}^{T}(y_t - Z_t'\theta)'\,\Sigma^{-1}(y_t - Z_t'\theta),$$

where Σ is the covariance matrix associated with the share vector y_t.

The "first-stage" of this procedure is to fit by single equation methods, finding $\bar{\theta}$ to minimize

$$S(\theta,I).$$

Σ is then estimated by the residuals in this stage, according to

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}(y_t - Z_t'\bar{\theta})(y_t - Z_t'\bar{\theta})'$$

and we then find the $\hat{\theta}$ which minimizes

$$S(\theta,\hat{\Sigma}).$$

In each stage, the minimization of S is constrained by imposing linear homogeneity in prices. This we accomplish by letting

$$r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \cdots \\ \underline{0} \end{bmatrix}, \quad R = \left[\begin{array}{ccc:c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underline{0} \\ 0 & 0 & 1 & \\ 0 & -1 & -1 & \\ \hdashline & \underline{0} & & \underline{I} \end{array}\right]$$

and then

$$\hat{\theta} = r + R\,(\sum_{t=1}^{T} R' Z_t \hat{\Sigma}^{-1} Z_t' R)^{-} \sum_{t=1}^{T} R' Z_t\, \hat{\Sigma}^{-1}(y_t - Z_t' r).$$ [9]

5. The Design of the Experiment

The central composite rotatable design is our choice of experimental design. As described by Cochran and Cox (1957), the design is used for studying two-dimensional response surfaces. We hypothesize, based on the previous work already summarized, that the two important parameters for our experiment are the Box-Cox parameter λ, and σ, a measure of magnitude in the substitution matrix.[10] The plan shown in Figure 2 is taken from Figure 8A.2 of Cochran and Cox (1957, page 346). Figure 3 is an affine transformation of Figure 2 and displays the value of σ and λ associated with the points of the classic design. Note that we orient the design so that the translog and Leontief technologies are extreme points: $(-\sqrt{2}, 0)$ and $(0, -\sqrt{2})$ respectively. Any departures away from those points are in the direction of greater dispersion in the substitution matrix and a larger Box-Cox parameter (thereby departing from the Fourier class). With this setup, we are able to analyze the performance of the Fourier flexible form over the entire surface.

The Box-Cox parameter λ varies from 0 to 2, covering the full range of the generalized Box-Cox; the same range for σ covers reasonable magnitudes and provides some convenient symmetry in the coordinates of the design points. Our expectations are that the range of σ is of less importance than that of λ, since the translog (and therefore the logarithmic Fourier) can involve any value for σ. Given the results of Wales (1977) and Guilkey, Lovell, and Sickles (1981), however, that the translog approximation deteriorates as σ departs from unity, and as the Allen-Uzawa partial elasticities of substitution depart from each other, this range does admit some potentially interesting variation in the chosen technologies.

We make the following assumptions, to characterize the matrix of substitution elasticities with just one parameter. The (compensated) own-price elasticities are assumed to be equal to $-\nu$ for all factors. Therefore, each diagonal element of Σ depends only on the share of the corresponding input, so that

$$\sigma_{ii} = (-\nu/S_i), \quad i=1,\ldots,n.$$

The off-diagonals, or partial elasticities of substitution, are then assumed to each have the same magnitude, σ. A negative sign is randomly allocated at each design point to one of the three off-diagonal terms (say, σ_{23}), to give us a lower triangle of the form

$$\begin{pmatrix} \sigma & \\ \sigma & -\sigma \end{pmatrix}.$$

This allows our experiment to consider technologies outside the constant elasticity of substitution (CES) class. The effect is that two of the relationships between factors represent substitutability, the third complementarity.

Given a price vector and then assigning the value of σ to that price vector, one finds that the rest of the substitution matrix is determined entirely by the placement of the negative sign because of the conditions imposed by "adding-up." Starting from the structure

$$\Sigma = \begin{vmatrix} -\nu/S_1 & & \\ \sigma & -\nu/S_2 & \\ \sigma & -\sigma & -\nu/S_3 \end{vmatrix}$$

THE CLASSICAL CENTRAL COMPOSITE LAYOUT

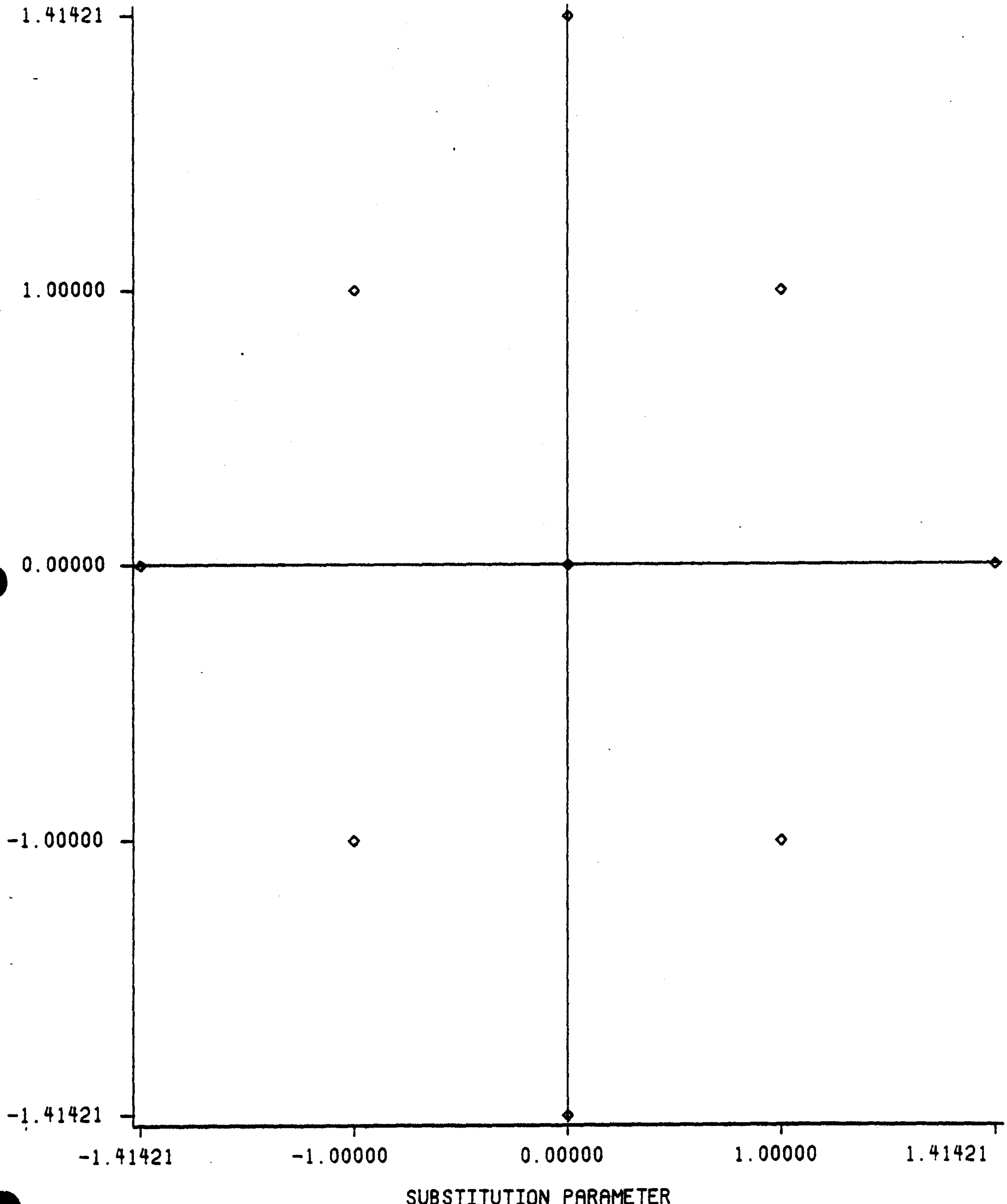

FIGURE 2

THE REORIENTED CENTRAL COMPOSITE DESIGN

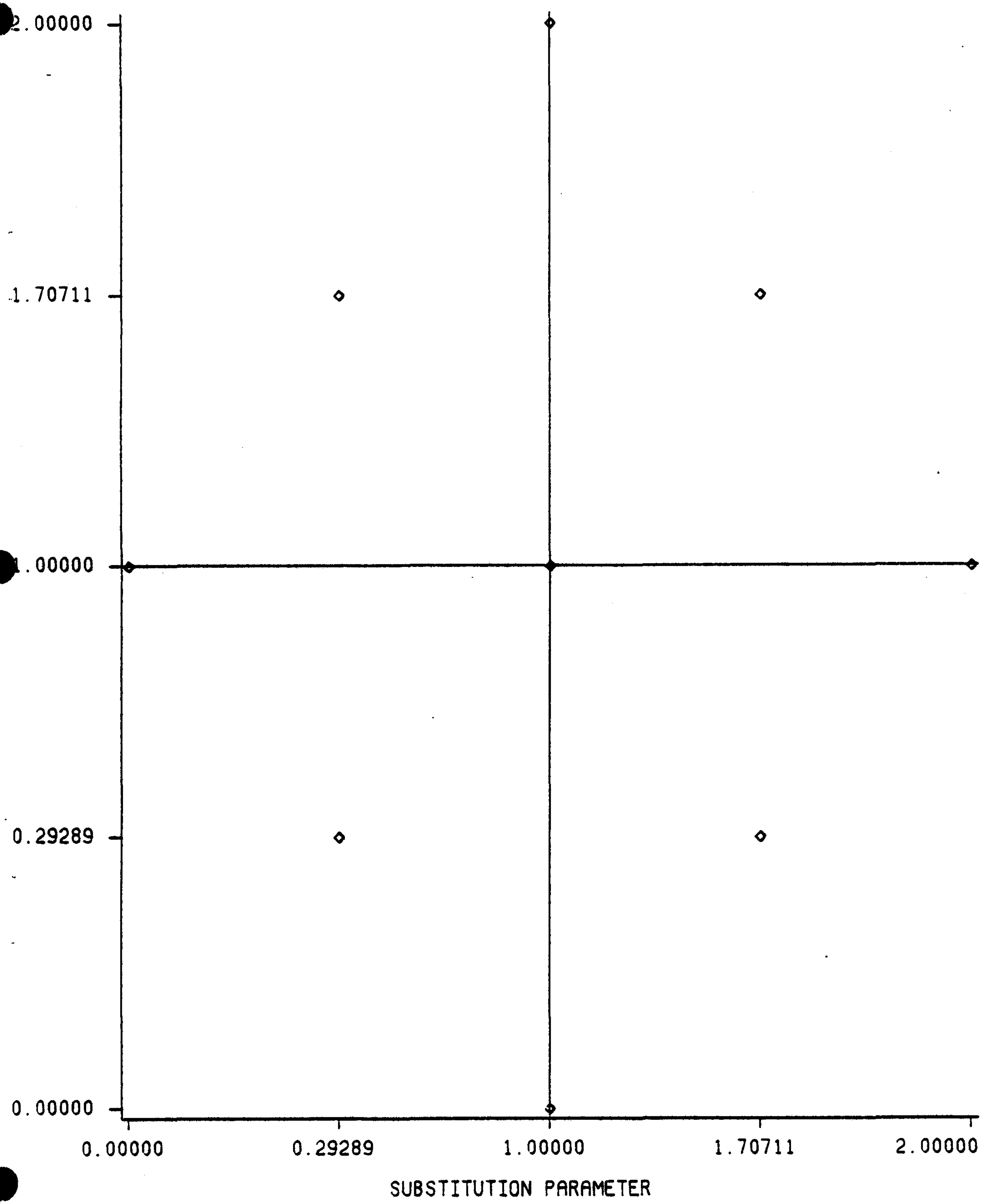

FIGURE 3

we know by one of the Euler conditions that

$$\sum_j \sigma_{ij} S_j = S_1(-\nu/S_1) + S_2\sigma + S_3\sigma = 0$$

Similarly,

$$S_1\sigma + S_2(-\nu/S_2) + S_3(-\sigma) = 0$$

and

$$S_1\sigma + S_2(-\sigma) + S_3(-\nu/S_3) = 0.$$

Solving these expressions simultaneously with

$$S_1 + S_2 + S_3 = 1,$$

we find that these four expressions hold only if

$$\nu = (2\sigma/5).$$

So, as σ goes from 0 to 2, the (compensated) own-price elasticity of factor demand goes from 0 to 0.8. Also, shares are determined by these conditions. The general rule is, for σ_{ij} negative, S_i and S_j are .2, while S_k is .6.[11] These results are used to solve for the parameter values of the various generalized Box-Cox cases, in order to provide technologies which yield these elasticities and shares at a certain price vector.

6. Experimental Data

We have used U. S. manufacturing data (Berndt and Wood (1975), Berndt and Khaled (1979)) to generate our experimental data. The time series these authors have used includes four factors: labor, capital, energy, and materials. To reduce to three goods, we aggregate the latter two inputs, weighted according to the share of each input in total costs for each period. We generate observations for this aggregate input A by

$$P_A = (S_e P_e + S_m P_m)/(S_e + S_m)$$

and

$$A = (P_e E + P_m M)/P_A.$$

The share of input A is then the sum of the energy and materials shares. We construct a price index of the form

$$P = TIC/Q$$

where TIC is total input cost and Q is output. We then are interested only in the three variables

$$X_1 = \log(P_A/P)$$

$$X_2 = \log\ (P_K/P)$$

and

$$X_3 = \log(P_L/P).$$

In order to create data sets of arbitrary length that are structured similarly to these data, we assume that the Berndt-Wood prices are generated by a first-order autoregressive process, such that the vector of logged real prices in period t is

$$P_t = \beta + U_t$$

with

$$U_t = RU_{t-1} + e_t,$$

where $e_t \sim N_3(0,S)$.

This process was fitted to the modified 1947-1971 data set. We find

$$\beta = (X_1,X_2,X_3)' = (.09760778, -.00721513, .37572201)',$$

$$S = \begin{bmatrix} 0.000162509 & 0.00011140 & -0.000014491 \\ & 0.00703273 & 0.000040986, \\ & & 0.000793996 \end{bmatrix},$$

and

$$R = \begin{bmatrix} 0.56627 & 0.055096 & -0.01754 \\ 0.13356 & 0.502859 & 0.129266 \\ -0.14791 & 0.110825 & 0.934661 \end{bmatrix}.$$

With these estimates in hand, one can create data sets of arbitrary length similar to that of Berndt and Wood (1975) as follows:[12] Generate normal (0,I) errors using the GGNML program of IMSL (1979). Then convert these to observations on a multivariate normal vector e_t with

$$e_t \sim N_3(0,S),$$

where S=FF, by pre-multiplication of vectors of length three by

$$F = \begin{bmatrix} -0.00135824 & -0.08386093 & -0.00054760 \\ 0.00067686 & 0.00017299 & -0.02817099 \\ -0.01265727 & 0.00206979 & -0.00030284 \end{bmatrix}.$$

Finally, apply the expression for the observed prices in each period

$$P_t = \beta + RU_{t-1} + e_t$$

to create a series of prices.

The parameters of the generalized Box-Cox cost function at each design point were chosen such that at the price

$$P = e^{\beta}$$

the elasticity of substitution matrix assigned to that design point obtained. To do this, one solves the nonlinear system

$$C = [(2/\lambda) \sum_i \sum_j \gamma_{ij} P_i^{\lambda/2} \; P_j^{\lambda/2}]^{1/\lambda},$$

$$S_i = 2P_i^{\lambda/2} (\sum_j \gamma_{ij} P_j^{\lambda/2})/\lambda \; C^{\lambda}, \quad i=1,\ldots,n,$$

$$\sigma_{ii} = 1 - \lambda + \gamma_{ii} \frac{P_i^{\lambda}}{S_i^2} C^{-\lambda} + \frac{\lambda}{2S_i} - \frac{1}{S_i}, \quad i=1,\ldots,n,$$

and

$$\sigma_{ij} = 1 - \lambda + \gamma_{ij} \frac{(P_i P_j)^{\lambda/2}}{S_i S_j} \; C^{-\lambda} \quad (i \neq j).$$

Figure 4 displays the results of these computations.

1. Translog: $\ell nC = \alpha_0 + \sum_i \alpha_i \ell n\ P_i + \frac{1}{2}\sum_i \sum_j \gamma_{ij}\ \ell nP_i\ \ell nP_j$

$\alpha_0 = 0.756928,\ \alpha_1 = 0.191614,\ \alpha_2 = 0.208386,\ \alpha_3 = 0.6$
$\gamma_{11} = 0.08,\ \gamma_{12} = -0.08,\ \gamma_{22} = 0.08,\ \gamma_{13} = \gamma_{23} = \gamma_{33} = 0$

2. Gen. Leontief: $C = (2 \sum_i \sum_j \gamma_{ij}\ P_i^{1/2}\ P_j^{1/2})$ $(\lambda=1,\ \sigma=1)$

$\gamma_{11} = 0.1088,\ \gamma_{12} = 0.2294,\ \gamma_{13} = 0.1894$
$\gamma_{22} = 0.0403,\ \gamma_{23} = -0.0665,\ \gamma_{33} = 0.0275$

3. $\lambda = 1.70711,\ \sigma = 1.70711$

$\gamma_{11} = -0.0667,\ \gamma_{12} = 0.3142,\ \gamma_{13} = 0.2266$
$\gamma_{22} = -0.0937,\ \gamma_{23} = -0.0342,\ \gamma_{33} = -0.0487$

4. Gen. Square Root Quadratic $(\lambda = 2,\ \sigma = 1)$

$\gamma_{11} = -0.0329,\ \gamma_{12} = 0.0,\ \gamma_{13} = 0.1495$
$\gamma_{22} = -0.0406,\ \gamma_{23} = 0.1660,\ \gamma_{33} = 0.0566$

5. $\lambda = 1.70711,\ \sigma = 0.29289$

$\gamma_{11} = 0.2699,\ \gamma_{12} = 0.1302,\ \gamma_{13} = 0.0939$
$\gamma_{22} = 0.0405,\ \gamma_{23} = 0.0142,\ \gamma_{33} = 0.0211$

6. $\lambda = 0.29289,\ \sigma = 0.29289$

$\gamma_{11} = 0.7896,\ \gamma_{12} = -0.2696,\ \gamma_{13} = -0.3167$
$\gamma_{22} = 0.8143,\ \gamma_{23} = -0.3216,\ \gamma_{33} = 1.1456$

7. $\lambda = 0.29289,\ \sigma = 1.70711$

$\gamma_{11} = -1.0095,\ \gamma_{12} = 0.8086,\ \gamma_{13} = 0.7645$
$\gamma_{22} = 0.0401,\ \gamma_{23} = -0.6248,\ \gamma_{33} = 0.0358$

8. Gen. Leontief $(\lambda = 1,\ \sigma = 2)$

$\gamma_{11} = -0.1088,\ \gamma_{12} = 0.4588,\ \gamma_{13} = -0.1263$
$\gamma_{22} = -0.3626,\ \gamma_{23} = 0.3992,\ \gamma_{33} = -0.0824$

9. Leontief $(\lambda = 1,\ \sigma = 0)$

$\gamma_{11} = 0.1814,\ \gamma_{22} = 0.6043,\ \gamma_{33} = 0.1374$
$\gamma_{12} = \gamma_{13} = \gamma_{23} = 0.0$

Figure 4: Technologies Assigned to Each Design Point

7. Errors in Variables: Expected and Actual Prices

A plausible scenario (Rossi, 1982) is that producers choose input and output quantities on the basis of expected prices $\hat{p}_t$. Ex-post one observes actual prices p_t which differ from expected prices by a random error v_t. The observed input shares s_t correspond to the unobservable $\hat{p}_t$. The regression of observed s_t on observed p_t by Seemingly Unrelated Regressions represents an error-in-variables problem.

One would be hard pressed to describe a scenario that would produce a data structure conforming to the formal assumptions of the Seemingly Unrelated Regressions method. Rossi (1982) discusses this point in detail. Implicit in the widespread use of the Seemingly Unrelated Regressions method in demand analysis must be an assumption that the errors-in-variables problem is not severe enough to matter. We accommodate this situation as follows.

In each period, the innovation U_t follows

$$U_t = RU_{t-1} + e_t$$

where e_t is the multivariate normal vector of the previous section. Then, actual prices are assumed to be

$$P_t = \beta + U_t$$

while expected prices follow

$$\hat{P}_t = P_t + V_t$$

where $V_t = \beta \cdot Y_t$ with $Y_t \sim N_3(0, .1 \cdot I_3)$.

Refer to the translog as a base case and recall that the logarithmic version of the Fourier form includes this form as a special case (all the sine/cosine terms have zero coefficients). We set the errors-in-variables problem and the variation induced in our prices over time at levels which allow reasonable accuracy in the translog case. The same environment is then used for the other eight technologies fitted with the Fourier form. Comparisons are therefore possible, over the points of our design, of the ability of the Fourier form to approximate each technology.

8. Monte Carlo Results

Guilkey, Lovell, and Sickles (1981) calculate the mean absolute deviation of their estimates for the various flexible forms they test. This involves averaging the absolute differences between true and estimated elasticities over the observations on prices in their simulations, and provides a convenient summary statistic. Since our experiment features elasticities calculated only at one point, the bias in our estimate is the same as the mean absolute deviation of Guilkey, Lovell, and Sickles (1981).

The tables in Appendix 1 summarize, for the nine design points, the means of our Monte Carlo estimates and true values for the six substitution elasticities of interest. We present first the $K_T = 13$ cases; these feature sample size T=25, replicated over 5000 trials.

Our next step was to estimate the same technologies for $K_T=22$ and (according to a chosen growth rule), T=48.[13] We find, however, that there is not a significant reduction in bias. Indeed, some of the parameters were estimated with slightly greater bias. This indicates that most of the bias present in the $K_T=13$ cases can be attributed to the errors-in-variables bias, with the specification error having been made very small by even a few Fourier parameters.

Using the results obtained from the Monte Carlo trials, we fit a response surface to the measured bias in our estimates of substitution elasticities. According to the central composite rotatable design, we regress the bias on a quadratic response in our two variables λ and σ. The results of fitted response curves are displayed in Appendix 2. Guilkey, Lovell, and Sickles (1981), Wales (1977), and others have evaluated the estimation by flexible forms of cross-partial elasticities of substitution. We present results from our Monte Carlo simulations both for all elasticities estimated and for the "cross-partials" only. For each case, we have regressed both the absolute bias and the absolute percentage bias on a quadratic in λ and σ, the parameters of our central composite design.

As is evident from the results displayed in Appendix 2, the response curves are somewhat flat in σ and λ. At least, the situation does not seem to indicate a dramatic decrease in the performance of the Fourier form. Our expectations were that if the surface were anything but flat, it would likely be increasing in λ, as this would measure departure from the translog (and therefore misspecification). While we do observe increases in predicted bias as λ becomes large, the situation does not appear to deteriorate dramatically. From the fitted quadratic response curves, one can see that the bias does not appear to become very large; the surface seems flat. Although the fit of the quadratic is not particularly good from the standpoint of significant coefficients, the lack-of-fit tests performed in all eight regressions indicate that the problem is "within-point" variation in fitted elasticities, rather than a surface which cannot be fit with a quadratic in λ and σ. One can see from the Monte Carlo results that this within-point variation does not seem to be systematic.

9. Summary and Conclusions

In a three-input case we have estimated a set of technologies with the logarithmic version of the Fourier flexible form cost function. Monte Carlo results indicate that, after setting the bias in estimated substitution elasticities at a reasonable level for the translog case, the degree of bias does not change much as we depart from the translog. Since the translog is a special case of the logarithmic Fourier form, while other technologies fitted are not, this indicates that this form is useful for a wide range of technologies. Bias appears to be due not to misspecification, but to errors in our price variables.

We see important implications. For technologies which are likely to be well-approximated by the generalized Box-Cox, the Fourier flexible form is successful in approximation. Given several advantages of the latter, it is a superior alternative to the generalized Box-Cox. Secondly, and more importantly, our results show that in analyses of production technologies, there is no reason to fit anything but a logarithmic Fourier form. The technolgies we have fitted include the commonly-used flexible forms, and the logarithmic Fourier form has approximated each at a level of error which is clearly acceptable in applications.

Our results provide evidence that the Fourier form may be used with even a small number of parameters, without inducing sizeable specification errors. As long as the situation is one in which other types of specification error (errors-in-variables, for instance) are not so devastating as to make estimation with a flexible form meaningless, it can be expected to outperform all commonly-used fixed-parameter forms (translog, generalized Leontief, etc.). The hypothesis that the Fourier flexible form is only a good idea in technologies resembling the translog, with increasing specification error as the translog is further from the truth, is rejected by our results.

The 13-parameter form has been shown in the presence of errors-in-variables to provide nearly unbiased estimates for a range of technologies. One is free in applications to fit more parameters when the data permit. Previous papers have illustrated the properties of global (Sobolev) flexibility, asymptotically unbiased and size α tests, and consistent estimation; results here indicate that for a typical number of parameters, the Fourier form behaves nicely. In view of previous work which indicates that fixed-parameter flexible forms will not accomplish the same degree of approximation, we see no reason to fit any other flexible form.

APPENDIX 1

Means of Monte Carlo Estimates: $K_T = 13$.

$\sigma=1$, $\lambda=0$

DESIGN POINT ONE
(Translog)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-2	-1.99851989	0.00173043	+0.0014801
σ_{12}	5000	-1	-1.00220683	0.00169691	-0.00220683
σ_{13}	5000	1	1.00023445	0.00014149	+0.00023445
σ_{22}	5000	-2	-1.99935197	0.00253645	+0.0006480
σ_{23}	5000	1	1.00052783	0.00061691	+0.00052783
σ_{33}	5000	-.6666	-0.66692274	0.00020918	-0.0002561

$\sigma=1$, $\lambda=1$

DESIGN POINT TWO
(Generalized Leontief)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-.6666	-0.66686550	0.00012525	-0.0001988
σ_{12}	5000	1	1.00025598	0.00038172	+0.00025598
σ_{13}	5000	1	1.00026432	0.00033843	+0.00026432
σ_{22}	5000	-2	-1.97893199	0.02830558	+0.0210681
σ_{23}	5000	-1	-1.03642765	0.02963974	-0.03642765
σ_{33}	5000	-2	-1.94360121	0.03723246	+0.0563988

$\sigma=1.70711$, $\lambda=1.70711$

DESIGN POINT THREE

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-1.13807	-1.14153691	0.00106713	-0.0034669
σ_{12}	5000	1.70711	1.70413064	0.00285015	-0.0029794
σ_{13}	5000	1.70711	1.71917575	0.00179385	+0.0120657
σ_{22}	5000	-3.41422	-3.41617236	0.01976116	-0.0019523
σ_{23}	5000	-1.70711	-1.69444985	0.01700956	+0.0126602
σ_{33}	5000	-3.41422	-3.46263018	0.01765935	-0.0484101

$\sigma=1$, $\lambda=2$

DESIGN POINT FOUR
(Generalized Square-Root Quadratic)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-2	-2.01187159	0.00285970	-0.01187
σ_{12}	5000	-1	-0.99982509	0.00171917	+0.00018
σ_{13}	5000	1	1.00395219	0.00077736	+0.00395
σ_{22}	5000	-2	-1.99428069	0.00687460	+0.00572
σ_{23}	5000	1	0.99780458	0.00215681	-0.00220
σ_{33}	5000	-.6666	-0.66721793	0.00077364	-0.00055

$\sigma=0.29289$, $\lambda=1.70711$

DESIGN POINT FIVE

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-0.19526	-0.19449171	0.00099703	+0.00077
σ_{12}	5000	0.29289	0.29341973	0.00119453	+0.00045
σ_{13}	5000	0.29289	0.28967853	0.00312843	-0.00321
σ_{22}	5000	-0.58578	-0.56504704	0.01830686	+0.02073
σ_{23}	5000	-0.29289	0.31854790	0.01922044	-0.02566
σ_{33}	5000	-0.58578	-0.54720311	0.02646510	-0.03858

$\sigma=0.29289$, $\lambda=0.29289$

DESIGN POINT SIX

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-0.58578	-0.60008659	0.00725947	-0.01431
σ_{12}	5000	-0.29289	-0.29569265	0.00352104	-0.00280
σ_{13}	5000	0.29289	0.29843250	0.00262177	+0.00554
σ_{22}	5000	-0.58578	-0.53660596	0.04605583	+0.04917
σ_{23}	5000	0.29289	0.28537908	0.00862959	-0.00751
σ_{33}	5000	-0.19526	-0.19527744	0.00281018	-0.00002

σ=1.70711, λ=0.29289 DESIGN POINT SEVEN

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-1.13807	-1.13900661	0.00093683	-0.00093
σ_{12}	5000	1.70711	1.71071391	0.00276631	+0.00360
σ_{13}	5000	1.70711	1.70672610	0.00115887	-0.00038
σ_{22}	5000	-3.41422	-3.42881709	0.01896029	-0.014597
σ_{23}	5000	-1.70711	-1.69972698	0.01631455	+0.00738
σ_{33}	5000	-3.41422	-3.42716223	0.01763584	-0.01294

σ=2, λ=1 DESIGN POINT EIGHT
(Generalized Leontief)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-4	-4.00013174	0.01038219	-0.00013
σ_{12}	5000	2	1.99920592	0.00329683	-0.00079
σ_{13}	5000	-2	-1.99703949	0.00402267	+0.00296
σ_{22}	5000	-1.3333	-1.33533043	0.00221755	-0.00200
σ_{23}	5000	2	2.00631372	0.00589196	+0.00631
σ_{33}	5000	-4	-4.02230158	0.01878602	-0.02230

σ=0, λ=1 DESIGN POINT NINE
(Leontief)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	0	-0.01179970	0.00693466	-0.01179
σ_{12}	5000	0	0.00085424	0.00165078	0.00085
σ_{13}	5000	0	0.00839825	0.00706013	0.00840
σ_{22}	5000	0	-0.00591069	0.00511371	-0.00591
σ_{23}	5000	0	0.01344565	0.01562308	0.01345
σ_{33}	5000	0	-0.04082633	0.05508760	-0.04083

MEANS OF MONTE CARLO ESTIMATES WITH $K_T = 22$

$\sigma=1, \lambda=0$

DESIGN POINT ONE
(Translog)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-2	-1.99515887	0.00124979	+0.00484
σ_{12}	5000	-1	-1.00382673	0.00112039	-0.00383
σ_{13}	5000	1	0.99966593	0.00016176	+0.00033
σ_{22}	5000	-2	-1.99866640	0.00131713	+0.00133
σ_{23}	5000	1	1.00081631	0.00024547	+0.00082
σ_{33}	5000	-.6666	-0.66682416	0.00008597	-0.00016

$\sigma=1, \lambda=1$

DESIGN POINT TWO
(Generalized Leontief)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-.6666	-0.66642939	0.00012870	+0.00024
σ_{12}	5000	1	0.99913564	0.00056276	-0.00086
σ_{13}	5000	1	1.00012969	0.00054956	+0.00013
σ_{22}	5000	-2	-1.99829392	0.01064387	+0.00171
σ_{23}	5000	-1	-1.00046690	0.01063388	-0.00047
σ_{33}	5000	-2	-1.99878873	0.01120892	+0.00121

$\sigma=1.70711, \lambda=1.70711$

DESIGN POINT THREE

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-1.13807	-1.13844172	0.00085888	-0.0003717
σ_{12}	5000	1.70711	1.70503529	0.00233526	-0.0020748
σ_{13}	5000	1.70711	1.71082237	0.00165366	+0.0037123
σ_{22}	5000	-3.41422	-3.40455996	0.01056636	+0.0096601
σ_{23}	5000	-1.70711	-1.71012598	0.00762072	-0.0030159
σ_{33}	5000	-3.41422	-3.42484569	0.00901968	-0.0106256

$\sigma=1$, $\lambda=2$

DESIGN POINT FOUR
(Generalized Square Root Quadratic)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-2	-2.00542183	0.00258027	-0.00542183
σ_{12}	5000	-1	-1.00516471	0.00186577	-0.00516471
σ_{13}	5000	1	1.00328455	0.00066141	+0.00328455
σ_{22}	5000	-2	-1.99343450	0.00312732	+0.0066566
σ_{23}	5000	1	0.99929709	0.00089895	-0.0007030
σ_{33}	5000	-.6666	-0.66737055	0.00040677	-0.0007039

$\sigma=0.29289$, $\lambda=1.70711$

DESIGN POINT FIVE

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-0.19526	-0.19518331	0.00077394	+0.0000767
σ_{12}	5000	0.29289	0.29498221	0.00170444	+0.0020922
σ_{13}	5000	0.29289	0.29044410	0.00271260	-0.0024459
σ_{22}	5000	-0.58578	-0.58613951	0.00913800	-0.0003595
σ_{23}	5000	-0.29289	-0.29924516	0.01005323	-0.0063551
σ_{33}	5000	-0.58578	-0.57123219	0.01567305	+0.0145479

$\sigma=0.29289$, $\lambda=0.29289$

DESIGN POINT SIX

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-0.58578	-0.59169745	0.00611421	-0.0059174
σ_{12}	5000	-0.29289	-0.28820892	0.00565599	+0.0046811
σ_{13}	5000	0.29289	0.29336741	0.00301707	+0.000477
σ_{22}	5000	-0.58578	-0.58788002	0.00995556	-0.0021
σ_{23}	5000	0.29289	0.29240219	0.00407014	-0.0004879
σ_{33}	5000	-0.19526	-0.19548328	0.00212500	-0.0002232

$\sigma=1.70711$, $\lambda=0.29289$

DESIGN POINT SEVEN

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-1.13807	-1.13779816	0.00078823	+0.0002719
σ_{12}	5000	1.70711	1.70560957	0.00222715	-0.0015005
σ_{13}	5000	1.70711	1.70799009	0.00136689	+0.000879
σ_{22}	5000	-3.41422	-3.42119718	0.01033958	-0.0069771
σ_{23}	5000	-1.70711	-1.69643752	0.00722716	+0.0106725
σ_{33}	5000	-3.41422	-3.42758948	0.00796646	-0.0133694

$\sigma=1$, $\lambda=1$

DESIGN POINT EIGHT
(Generalized Leontief)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	-4	-4.00172708	0.00877313	-0.00172708
σ_{12}	5000	2	2.00187013	0.00256611	+0.00187013
σ_{13}	5000	-2	-2.00427133	0.00487244	-0.00427133
σ_{22}	5000	-1.3333	-1.33375191	0.00119964	-0.0004186
σ_{23}	5000	2	1.99980450	0.00255049	-0.0001955
σ_{33}	5000	-4	-3.99646526	0.00885382	+0.0035348

$\sigma=0$, $\lambda=1$

DESIGN POINT NINE
(Leontief)

var	n	true value	mean estimate	standard error of the mean	bias
σ_{11}	5000	0	0.00199572	0.00684097	0.00199572
σ_{12}	5000	0	-0.00043965	0.00249818	-0.00043965
σ_{13}	5000	0	-0.00102451	0.00870260	-0.00102451
σ_{22}	5000	0	0.00117429	0.00207352	0.00117429
σ_{23}	5000	0	-0.00320273	0.00667987	-0.00320273
σ_{33}	5000	0	0.01070212	0.02512006	0.0107212

APPENDIX 2

Response Surfaces for Predicted Bias

1. ABSOLUTE BIASES, CROSS PARTIAL SUBSTITUTION ELASTICITIES*

Thirteen Parameter Fits

$$\begin{array}{lcccccc} \text{Bias} = & -0.00194 & + 0.01959\lambda & + 0.00831\sigma & - 0.00901\lambda^2 & - 0.00509\sigma^2 & + 0.00047\sigma\lambda \\ & (.0086) & (.0129) & (.0129) & (.0058) & (.0058) & (.0050) \end{array}$$

$R^2 = 0.1397$ SSPE = 0.00145, 18 d.f. $F_{LF} = 0.44$

Twenty-Two Parameter Fits

$$\begin{array}{lcccccc} \text{Bias} = & 0.00237 & - 0.00285\lambda & - 0.00177\sigma & + 0.00242\lambda^2 & + 0.00190\sigma^2 & - 0.00158\sigma\lambda \\ & (.0024) & (.0036) & (.0036) & (.0016) & (.0016) & (.0014) \end{array}$$

$R^2 = 0.1697$ SSPE = 0.000115 $F_{LF} = 0.46$

2. ABSOLUTE BIASES, ALL SIX SUBSTITUTION ELASTICITIES

Thirteen Parameter Fits

$$\begin{array}{lcccccc} \text{Bias} = & 0.00271 & + 0.02690^{*}\lambda & + 0.00691\sigma & - 0.01360^{**} & - 0.00644\sigma^2 & + 0.00263\sigma\lambda \\ & (.0099) & (.0148) & (.0148) & (.0068) & (.0068) & (.0057) \end{array}$$

*In this appendix, F_{LF} denotes the calculated F-statistic for lack-of-fit tests, SSPE the "pure" error sum of squares, and one, two, and three asterisks denote significance levels of 10%, 5% and 1%, respectively. Standard errors of parameter estimates are in parentheses. The regressions for absolute bias include all nine design points; the percentage bias results do not include the ninth design point, as percentages are meaningless for the Leontief case.

$R^2 = 0.1208 \quad SSPE = 0.00888, 45\text{ d.f.} \quad F_{LF} = 0.77$

Twenty-Two Parameter Fits

$$\underset{}{Bias} = \underset{(.0025)}{0.00374} - \underset{(.0038)}{0.00360\lambda} - \underset{(.0038)}{0.00342\sigma} + \underset{(.0017)}{0.00281\lambda^2} + \underset{(.0017)}{0.00259\sigma^2} - \underset{(.0015)}{0.00135\sigma\lambda}$$

$R^2 = 0.0949 \quad SSPE = 0.0005653, 45\text{ d.f.} \quad F_{LF} = 1.46$

3. PERCENTAGE BIAS, CROSS PARTIAL ELASTICITY OF SUBSTITUTION

Thirteen Parameter Fits

$$Bias = \underset{(.0197)}{0.02127} + \underset{(.0270)}{0.02760\lambda} - \underset{(.0326)}{0.03177\sigma} - \underset{(.0122)}{0.00896\lambda^2} + \underset{(.0136)}{0.01028\sigma^2} - \underset{(.0103)}{0.00611\sigma\lambda}$$

$R^2 = 0.3056 \quad SSPE = 0.00548, 16\text{ d.f.} \quad F_{LF} = 0.33$

Twenty-Two Parameter Fits

$$Bias = \underset{(.0047)}{0.111^{**}} - \underset{(.0064)}{0.0000796\lambda} - \underset{(.0075)}{0.01666^{**}\sigma} + \underset{(.0029)}{0.00237\lambda^2} + \underset{(.0032)}{0.00721^{**}\sigma^2} - \underset{(.0024)}{0.00340\sigma\lambda}$$

$R^2 = 0.4976 \quad SSPE = 0.0003, 16\text{ d.f.} \quad F_{LF} = 0.33$

4. PERCENTAGE BIAS, ALL SIX SUBSTITUTION ELASTICITIES

Thirteen Parameter Fits

$$Bias = \underset{(.0135)}{0.03453^{**}} + \underset{(.0185)}{0.02216\lambda} - \underset{(.0216)}{0.04736^{**}\sigma} - \underset{(.0084)}{0.00891\lambda^2} + \underset{(.0093)}{0.01429\sigma^2} - \underset{(.0070)}{0.00204\sigma\lambda}$$

$R^2 = 0.3453$ SSPE = 0.0122, 40 d.f. $F_{LF} = 0.57$

Twenty-Two Parameter Fits

$$\text{Bias} = \underset{(.0034)}{0.00964^{***}} - \underset{(.0047)}{0.00031\lambda} - \underset{(.0055)}{0.01424^{**}\sigma} + \underset{(.0021)}{0.00202\lambda^2} + \underset{(.0024)}{0.00605\sigma^2} - \underset{(.0018)}{0.00261\sigma\lambda}$$

$R^2 = 0.3536$ SSPE = 0.00079, 40 d.f. $F_{LF} = 0.62$

Footnotes

[1]The generalized Box-Cox is a generalization of functional forms used by Denny (1974) and Kiefer (1976). Denny, in a production study, specifies the "generalized quadratic" functional form. This gives the cost function

$$C = \left(\sum_i \sum_j b_{ij} P_i^{\beta\gamma} P_j^{\beta(1-\gamma)}\right)^{1/\beta} h(Y)$$

which, with $\gamma = \frac{1}{2}$, is the form we use to generate data. Kiefer's indirect utility function involves a Box-Cox transformation (Box and Cox (1964) of the normalized prices (Pi/m). This is accomplished by

$$f(P_i/m) = \frac{(P_i/m)^{\lambda} - 1}{\lambda}$$

with $\lambda \to 0$ causing f(•) to behave as the natural log. That is, the limit of the indirect-utility function as $\lambda \to 0$ is a translog indrect utility function (Christensen, Jorgensen, and Lau (1975)).

[2]Substitution elasticities must not oscillate wildly over the region of interest and the number of fitted parameters is allowed to increase as the number of observations increases. Any of the common estimation procedures (multivariate least-squares, maximum likelihood, and three-stage least-squares) produce consistency if these conditions are met. The interested reader should consult El Badawi, Gallant, and Souza (1982) for details.

[3]See Gallant (1982) for a discussion of "Sobolev flexibility" as opposed to "Diewert flexibility" and a demonstration that the Fourier form has the former characteristic.

[4]From each term in the series of logged prices for input i, we have subtracted the smallest observation, and added an adjustment factor, 10^{-5}. Each series is thereby adjusted so that 10^{-5} is the smallest observed price of each input. The same process would be applied to output. We therefore have $\ln a_i = -\min(X_i) + 10^{-5}$.

[5]In our hypothetical technologies, we have not attempted to equate total costs across the design, since substitution elasticities are invariant to changes in the measure of costs. Convenient normalization rules were used to obtain parameters in the generalized Box-Cox cost function which produced the desired substitution matrix; these produce different values for C at the same prices. Since interest centers only upon relative prices, this is of no consequence.

[6]Gallant (1982) uses vectors of dimension n + 1, in order to include output in the expression for the cost function, as element n + 1 of x, for n inputs. Since interest here is only in the n prices, the definition of multi-indexes reflects this. The choice of multi-indexes has been computerized, as

it becomes tedious for large numbers of inputs; see Monahan (1981) for FORTRAN code to handle this process.

[7] U_0 is zero because we fit only shares, not total costs.

[8] One share is deleted, as is customary when the system is singular in n shares; $f(x_t|\theta)$ is linear in the parameters and Zt is a 2 by 13 (or 22) matrix used to represent this fact.

[9] The generalized inverse is that described by Goodnight (1979), and is used to automatically adjust for the overparameterization of the matrix C. C has only three free parameters, and the generalized inverse automatically identifies that subset of $(u_{01}, u_{02}, \ldots, u_{0A})$ to be set to zero. See Gallant (1982) for details.

[10] By restricting the form of Σ, the substitution matrix, we are able to characterize it with one parameter. Other forms would be no less interesting, but would complicate our response surface, increase the number of design points, and contribute little to understanding the behavior of the Fourier form. Details of the structure of the substitution matrix are given below. See Appendix 1 for the placement of the complementarity relationship in each technology.

[11] For the $\lambda=1$, $\sigma=0$ case, we obtain a Leontief technology. In that case, the minus sign is used to choose which share is the larger of the three.

[12] The matrix R has eigenvalues 0.9662, 0.40966, and 0.6278, indicating that the autoregressive process is not an explosive one.

[13] El Badawi, Gallant, and Souza (1982) show that whether parameters grow at a deterministic rate, or are chosen adaptively, consistency obtains. We allow

$$\lim_{T\to\infty} \frac{K_T}{T} \to 0 \quad \text{but} \quad \lim_{T\to\infty} K_T \to \infty$$

by choosing

$$K_T = T^{.8}$$

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