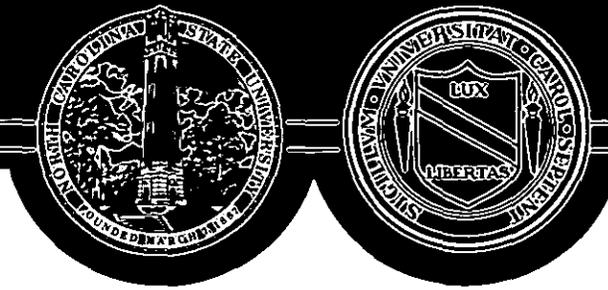


# THE INSTITUTE OF STATISTICS

UNIVERSITY OF NORTH CAROLINA SYSTEM



## Nonlinear Statistical Models

by  
A. Ronald Gallant

Chapter 4. Univariate Nonlinear Regression:  
Asymptotic Theory

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CHAPTER 4. Univariate Nonlinear Regression: Asymptotic Theory

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## CHAPTER 4. Univariate Nonlinear Regression: Asymptotic Theory

In this chapter, the results of the previous chapter are specialized to the case of a correctly specified univariate nonlinear regression model estimated by least squares. Specialization is simply a matter of restating Assumptions 1 through 7 of Chapter 3 in context. This done, the asymptotic theory follows immediately. The characterizations used in Chapter 1 are established using probability bounds that follow from the asymptotic theory.

## 1. INTRODUCTION

Let us review some notation. The univariate nonlinear model is written as

$$y_t = f(x_t, \theta^\circ) + e_t \quad t = 1, 2, \dots, n$$

with  $\theta^\circ$  known to lie in some compact set  $\Theta^*$ . The functional form of  $f(x, \theta)$  is known,  $x$  is  $k$ -dimensional,  $\theta$  is  $p$ -dimensional, and the model is assumed to be correctly specified. Following the conventions of Chapter 1, the model can be written in a vector notation as

$$y = f(\theta^\circ) + e$$

with the Jacobian of  $f(\theta)$  written as  $F(\theta) = (\partial/\partial\theta')f(\theta)$ . The parameter  $\theta$  is estimated by  $\hat{\theta}$  that minimizes

$$s_n(\theta) = (1/n) \|y - f(\theta)\|^2 = (1/n) \sum_{t=1}^n [y_t - f(x_t, \theta)]^2.$$

We are interested in testing the hypothesis

$$H: h(\theta^\circ) = 0 \quad \text{against} \quad A: h(\theta^\circ) \neq 0$$

which we assume can be given the equivalent representation

$$H: \theta^\circ = g(\rho^\circ) \text{ for some } \rho^\circ \text{ against } A: \theta^\circ \neq g(\rho) \text{ for any } \rho$$

where  $h: \mathbb{R}^p \rightarrow \mathbb{R}^q$ ,  $g: \mathbb{R}^r \rightarrow \mathbb{R}^p$ , and  $p = r + q$ . The correspondence with the notation of Chapter 3 is as follows.

## NOTATION 1

General (Chapter 3)

$$e_t = q(y_t, x_t, \gamma_n^\circ)$$

$$\gamma \in \Gamma$$

$$y = Y(e, x, \gamma)$$

$$s(y_t, x_t, \hat{\tau}_n, \lambda)$$

$$\lambda \in \Lambda^*$$

$$s_n(\lambda) = (1/n) \sum_{t=1}^n s(y_t, x_t, \hat{\tau}_n, \lambda)$$

$$s_n^\circ(\lambda) = (1/n) \sum_{t=1}^n \int_{\mathcal{E}} s[Y(e, x_t, \gamma_n^\circ), x_t, \tau_n^\circ, \lambda] dP(e)$$

$$s^*(\lambda) = \int_{\mathcal{X}} \int_{\mathcal{E}} s[Y(e, x, \gamma^*), x, \tau^*, \lambda] dP(e) d\mu(x)$$

$$\hat{\lambda}_n \text{ minimizes } s_n(\lambda)$$

$$\tilde{\lambda}_n \text{ minimizes } s_n(\lambda)$$

$$\text{subject to } h(\lambda) = 0$$

$$\lambda_n^\circ \text{ minimizes } s_n^\circ(\lambda)$$

$$\lambda_n^* \text{ minimizes } s_n^\circ(\lambda)$$

$$\text{subject to } h(\lambda) = 0$$

$$\lambda^* \text{ minimizes } s^*(\lambda)$$

Specific (Chapter 4)

$$e_t = y_t - f(x_t, \theta_n^\circ)$$

$$\theta \in \Theta^*$$

$$y = f(x, \theta) + e$$

$$[y_t - f(x_t, \theta)]^2$$

$$\theta \in \Theta^*$$

$$s_n(\theta) = (1/n) \sum_{t=1}^n [y_t - f(x_t, \theta)]^2$$

$$s_n^\circ(\theta) = \sigma^2 + (1/n) \sum_{t=1}^n [f(x_t, \theta_n^\circ) - f(x_t, \theta)]^2$$

$$s^*(\theta) = \sigma^2 + \int_{\mathcal{X}} [f(x, \theta^*) - f(x, \theta)]^2 d\mu(x)$$

$$\hat{\theta}_n \text{ minimizes } s_n(\theta)$$

$$\tilde{\theta}_n = g(\hat{\rho}_n) \text{ minimizes } s_n(\theta)$$

$$\text{subject to } h(\theta) = 0$$

$$\theta_n^\circ \text{ minimizes } s_n^\circ(\theta)$$

$$\theta_n^* = g(\rho_n^\circ) \text{ minimizes } s_n^\circ(\theta)$$

$$\text{subject to } h(\theta) = 0$$

$$\theta^* \text{ minimizes } s^*(\theta)$$

## 2. REGULARITY CONDITIONS

Application of the general theory to a correctly specified univariate nonlinear regression is just a matter of restating Assumptions 1 through 7 of Chapter 3 in terms of the notation above. As the data is presumed to be generated according to

$$y_t = f(x_t, \theta_n^0) + e_t \quad t = 1, 2, \dots, n$$

Assumptions 1 through 5 of Chapter 3 read as follows.

ASSUMPTION 1'. The errors are independently and identically distributed with common distribution  $P(e)$ .  $\square$

ASSUMPTION 2'.  $f(x, \theta)$  is continuous on  $\mathcal{X} \times \Theta^*$  and  $\Theta^*$  is compact.  $\square$

ASSUMPTION 3'. (Gallant and Holly, 1980) Almost every realization of  $\{v_t\}$  with  $v_t = (e_t, x_t)$  is a Cesaro sum generator with respect to the product measure

$$\nu(A) = \int_{\mathcal{X}} \int_{\mathcal{E}} I_A(e, x) dP(e) d\mu(x)$$

and dominating function  $b(e, x)$ . The sequence  $\{x_t\}$  is a Cesaro sum generator with respect to  $\mu$  and  $b(x) = \int_{\mathcal{E}} b(e, x) dP(e)$ . For each  $x \in \mathcal{X}$  there is a neighborhood  $N_x$  such that  $\int_{\mathcal{E}} \sup_{N_x} b(e, x) dP(e) < \infty$ .  $\square$

ASSUMPTION 4'. (Identification) The parameter  $\theta^0$  is indexed by  $n$  and the sequence  $\{\theta_n^0\}$  converges to  $\theta^*$ .

$$s^*(\theta) = \sigma^2 + \int_{\mathcal{X}} [f(x, \theta^*) - f(x, \theta)]^2 d\mu(x)$$

has a unique minimum over  $\Theta^*$  at  $\theta^*$ .  $\square$

ASSUMPTION 5'.  $\Theta^*$  is compact,  $[e + f(x, \theta^0) - f(x, \theta)]^2$  is dominated by  $b(e, x)$ ;  $b(e, x)$  is that of Assumption 3.  $\square$

The sample objective function is

$$s_n(\theta) = (1/n) \|y - f(\theta)\|^2$$

with expectation

$$s_n^o(\theta) = \sigma^2 + (1/n) \|f(\theta_n^o) - f(\theta)\|^2.$$

By Lemma 1 of Chapter 3, both  $s_n(\theta)$  and  $s_n^o(\theta)$  have uniform, almost sure limit

$$s^*(\theta) = \sigma^2 + \int_{\mathcal{X}} [f(x, \theta^*) - f(x, \theta)]^2 d\mu(x).$$

Note that the true value  $\theta_n^o$  of the unknown parameter is also a minimizer of  $s_n^o(\theta)$  so that our use of  $\theta_n^o$  to denote them both is not ambiguous. We may apply Theorem 3 of Chapter 3 and conclude that

$$\lim_{n \rightarrow \infty} \theta_n^o = \theta^*,$$

$$\lim_{n \rightarrow \infty} \hat{\theta}_n = \theta^* \text{ almost surely.}$$

Assumption 6 of Chapter 3 may be restated as follows.

ASSUMPTION 6!  $\Theta^*$  contains a closed ball  $\Theta$  centered at  $\theta^*$  with finite, nonzero radius such that

$$(\partial/\partial\theta_i) s[Y(e, x, \theta^o), x, \theta] = -2[e + f(x, \theta^o) - f(x, \theta)] (\partial/\partial\theta_i) f(x, \theta)$$

$$(\partial^2/\partial\theta_i \partial\theta_j) s[Y(e, x, \theta^o), x, \theta] = 2[(\partial/\partial\theta_i) f(x, \theta)] [(\partial/\partial\theta_j) f(x, \theta)]$$

$$- 2[e + f(x, \theta^o) - f(x, \theta)] (\partial^2/\partial\theta_i \partial\theta_j) f(x, \theta)$$

$$\{(\partial/\partial\theta_i) s[Y(e, x, \theta^o), x, \theta]\} \{(\partial/\partial\theta_j) s[Y(e, x, \theta^o), x, \theta]\}$$

$$= 4[e + f(x, \theta^o) - f(x, \theta)]^2 [(\partial/\partial\theta_i) f(x, \theta)] [(\partial/\partial\theta_j) f(x, \theta)]$$

are continuous and dominated by  $b(e, x)$  on  $\mathcal{E} \times \mathcal{X} \times \Theta^* \times \Theta$  for  $i, j = 1, 2, \dots, p$ .

Moreover,

$$J^* = 2 \int_{\mathcal{X}} [(\partial/\partial\theta) f(x, \theta^*)] [(\partial/\partial\theta) f(x, \theta^*)]' d\mu(x)$$

is nonsingular.  $\square$

Define

## NOTATION 2

$$Q = \int_{\mathcal{X}} [(\partial/\partial\theta)f(x, \theta^*)] [(\partial/\partial\theta)f(x, \theta^*)]' d\mu(x),$$

$$Q_n^{\circ} = (1/n) F'(\theta_n^{\circ}) F(\theta_n^{\circ}),$$

$$Q_n^* = (1/n) F'(\theta_n^*) F(\theta_n^*). \quad \square$$

Direct computation according to Notations 2 and 3 of Chapter 3 yields

(Problem 1).

$$\mathcal{J}^* = 4\sigma^2 Q$$

$$\mathcal{J}^* = 2 Q$$

$$u^* = 0$$

$$\mathcal{J}_n^{\circ} = 4\sigma^2 Q_n^{\circ}$$

$$\mathcal{J}_n^{\circ} = 2 Q_n^{\circ}$$

$$u_n^{\circ} = 0$$

$$\mathcal{J}_n^* = 4\sigma^2 Q_n^*$$

$$\mathcal{J}_n^* = 2 Q_n^* - (2/n) \sum_{t=1}^n [f(x_t, \theta_n^{\circ}) - f(x_t, \theta_n^*)] (\partial^2/\partial\theta\partial\theta') f(x_t, \theta_n^*)$$

$$u_n^* = (4/n) \sum_{t=1}^n [f(x_t, \theta_n^{\circ}) - f(x_t, \theta_n^*)]^2 [(\partial/\partial\theta)f(x_t, \theta_n^*)] [(\partial/\partial\theta) f(x_t, \theta_n^*)]'$$

Noting that

$$(\partial/\partial\theta)s_n(\theta) = (-2/n)F'(\theta)[e + f(\theta_n^\circ) - f(\theta)]$$

we have from Theorem 4 of Chapter 3 that

$$(1/\sqrt{n})F'(\theta_n^\circ)e \xrightarrow{\mathcal{L}} N(0, \sigma^2 Q)$$

and from Theorem 5 that

$$\sqrt{n}(\hat{\theta}_n - \theta_n^\circ) \xrightarrow{\mathcal{L}} N(0, \sigma^2 Q^{-1})$$

$$\lim_{n \rightarrow \infty} Q_n^\circ = Q.$$

The Pitman drift assumption is restated as follows.

ASSUMPTION 7'. (Pitman drift) The sequence  $\theta_n^\circ$  is chosen such that  $\lim_{n \rightarrow \infty} \sqrt{n}(\theta_n^\circ - \theta_n^*) = \Delta$ . Moreover,  $h(\theta^*) = 0$ .

Noting that

$$(\partial/\partial\theta)s_n^\circ(\theta) = (-2/n)F'(\theta)[f(\theta_n^\circ) - f(\theta)]$$

we have from Theorem 6 that

$$\lim_{n \rightarrow \infty} \tilde{a}_n = a^* \text{ almost surely}$$

$$\lim_{n \rightarrow \infty} \theta_n^* = \theta^*$$

$$\lim_{n \rightarrow \infty} Q_n^* = Q$$

$$(1/\sqrt{n})F'(\theta_n^*)e \xrightarrow{\mathcal{L}} N(0, \sigma^2 Q)$$

$$\lim_{n \rightarrow \infty} (1/\sqrt{n})F'(\theta_n^*)[f(\theta_n^\circ) - f(\theta_n^*)] = Q\Delta.$$

Assumption 13 of Chapter 3 is restated as follows.

ASSUMPTION 13'. The function  $h(\theta)$  is a once continuously differentiable mapping of  $\Theta$  into  $\mathbb{R}^q$ . Its Jacobian  $H(\theta) = (\partial/\partial\theta')h(\theta)$  has full rank ( $=q$ ) at  $\theta = \theta^*$ .  $\square$

## PROBLEMS

1. Use the derivatives given in Assumption 6 to compute  $\bar{\mathcal{J}}(\theta)$ ,  $\bar{\mathcal{J}}'(\theta)$ ,  $\bar{u}(\theta)$  and  $\bar{\mathcal{J}}(\theta)$ ,  $\bar{\mathcal{J}}'(\theta)$ ,  $\bar{u}(\theta)$  as defined in Notations 2 and 3 of Chapter 3.

## 3. CHARACTERIZATIONS OF LEAST SQUARES ESTIMATORS AND TEST STATISTICS

The first of the characterizations appearing in Chapter 1 is

$$\hat{\theta}_n = \theta_n^\circ + [F'(\theta_n^\circ) F(\theta_n^\circ)]^{-1} F'(\theta_n^\circ) e + o_p(1/\sqrt{n}) .$$

It is derived using the same sort of arguments as used in the proof of Theorem 5 of Chapter 3 so we shall be brief here; one can look at Theorem 5 for details. By Lemma 2 of Chapter 3 we may assume without loss of generality that  $\hat{\theta}_n$  and  $\theta_n^\circ$  are in  $\Theta$  and that  $(\partial/\partial\theta) s_n(\hat{\theta}_n) = o_p(1/\sqrt{n})$ . Recall that  $Q_n^\circ = Q + o(1)$  whence  $\mathcal{J}_n^\circ = \mathcal{J}^* + o(1)$ . By Taylor's theorem

$$\sqrt{n} (\partial/\partial\theta) s_n(\theta_n^\circ) = \sqrt{n} (\partial/\partial\theta) s_n(\hat{\theta}_n) + \bar{\mathcal{J}} \sqrt{n} (\theta_n^\circ - \hat{\theta}_n)$$

where  $\bar{\mathcal{J}} = \mathcal{J}^* + o_s(1)$ . Then

$$[\mathcal{J}^* + o_s(1)] \sqrt{n} (\hat{\theta}_n - \theta_n^\circ) = - \sqrt{n} (\partial/\partial\theta) s_n(\theta_n^\circ) + o_s(1)$$

which can be rewritten as

$$\mathcal{J}_n^\circ \sqrt{n} (\hat{\theta}_n - \theta_n^\circ) = - \sqrt{n} (\partial/\partial\theta) s_n(\theta_n^\circ) - [\mathcal{J}^* - \mathcal{J}_n^\circ + o_s(1)] \sqrt{n} (\hat{\theta}_n - \theta_n^\circ) + o_s(1) .$$

Now  $[\mathcal{J}^* - \mathcal{J}_n^\circ + o_s(1)] = o_s(1)$  and  $\sqrt{n} (\hat{\theta}_n - \theta_n^\circ) \xrightarrow{\mathcal{L}} N(0, \sigma^2 Q)$  which implies that  $\sqrt{n} (\hat{\theta}_n - \theta_n^\circ) = O_p(1)$  whence  $[\mathcal{J}^* - \mathcal{J}_n^\circ + o_s(1)] \sqrt{n} (\hat{\theta}_n - \theta_n^\circ) = o_p(1)$ . Thus we have that

$$\mathcal{J}_n^\circ \sqrt{n} (\hat{\theta}_n - \theta_n^\circ) = \sqrt{n} (\partial/\partial\theta) s_n(\theta_n^\circ) + o_p(1) .$$

There is an  $N$  such that for  $n \geq N$  the inverse of  $\mathcal{J}_n^\circ$  exists whence

$$\sqrt{n} (\hat{\theta}_n - \theta_n^\circ) = -\sqrt{n} (\mathcal{J}_n^\circ)^{-1} (\partial/\partial\theta) s_n(\theta_n^\circ) + o_p(1)$$

or

$$\hat{\theta}_n = \theta_n^\circ - (\mathcal{J}_n^\circ)^{-1}(\partial/\partial\theta)s_n(\theta_n^\circ) + o_p(1/\sqrt{n}) .$$

Finally,  $-(\mathcal{J}_n^\circ)^{-1}(\partial/\partial\theta)s_n(\theta_n^\circ) = [F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ)e$  which completes the argument.

The next characterization that needs justification is

$$s^2 = e'\{I - F(\theta_n^\circ)[F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ)\}e/(n-p) + o_p(1/n) .$$

The derivation is similar to the arguments used in the proof of Theorem 15 of Chapter 3; again we shall be brief and one can look at the proof of Theorem 15 for details. By Taylor's theorem

$$\begin{aligned} n[s_n(\theta_n^\circ) - s_n(\hat{\theta}_n)] &= n[(\partial/\partial\theta)s_n(\hat{\theta}_n)]'(\hat{\theta}_n - \theta_n^\circ) \\ &\quad + (n/2)(\hat{\theta}_n - \theta_n^\circ)'[(\partial^2/\partial\theta\partial\theta')s_n(\hat{\theta}_n)](\hat{\theta}_n - \theta_n^\circ) \\ &= n o_s(1/\sqrt{n})(\hat{\theta}_n - \theta_n^\circ) + (n/2)(\hat{\theta}_n - \theta_n^\circ)'[\mathcal{J}_n^\circ + o_s(1)](\hat{\theta}_n - \theta_n^\circ) \\ &= (n/2)(\hat{\theta}_n - \theta_n^\circ)'\mathcal{J}_n^\circ(\hat{\theta}_n - \theta_n^\circ) + o_p(1) . \end{aligned}$$

From the proceeding result we have

$$(\hat{\theta}_n - \theta_n^\circ) = [F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ)e + o_p(1/\sqrt{n})$$

whence

$$n[s_n(\theta_n^\circ) - s_n(\hat{\theta}_n)] = n e'F(\theta_n^\circ)[F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ)e + o_p(1) .$$

This equation reduces to

$$\|y - f(\hat{\theta})\|^2 = e'\{I - F(\theta_n^\circ)[F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ)\}e + o_p(1/n)$$

which completes the argument.

Next we show that

$$h(\hat{\theta}_n) = h(\theta_n^\circ) + H(\theta_n^\circ)[F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ)e + o_p(1/\sqrt{n}) .$$

A straightforward argument using Taylor's theorem yields

$$\sqrt{n} h(\hat{\theta}_n) = \sqrt{n} h(\theta_n^\circ) + \bar{H} \sqrt{n} (\hat{\theta}_n - \theta_n^\circ)$$

where  $\bar{H}$  has rows  $(\partial/\partial\theta')$   $h(\bar{\theta}_i)$  with  $\bar{\theta}_i = \lambda_i \hat{\theta}_n + (1-\lambda_i)\theta_n^\circ$  for some  $\lambda_i$  with  $0 \leq \lambda_i \leq 1$  whence

$$\sqrt{n} h(\hat{\theta}_n) = \sqrt{n} h(\theta_n^\circ) + [H(\theta_n^\circ) + o_s(1)] \sqrt{n} (\hat{\theta}_n - \theta_n^\circ) .$$

Since  $\sqrt{n} (\hat{\theta}_n - \theta_n^\circ)$  is bounded in probability we have

$$\begin{aligned} \sqrt{n} h(\hat{\theta}_n) &= \sqrt{n} h(\theta_n^\circ) + \sqrt{n} H(\theta_n^\circ)(\hat{\theta}_n - \theta_n^\circ) + o_s(1) \\ &= \sqrt{n} h(\theta_n^\circ) + H(\theta_n^\circ) \sqrt{n} \{[F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ)e + o_p(1/\sqrt{n})\} + o_s(1) \\ &= \sqrt{n} h(\theta_n^\circ) + \sqrt{n} H(\theta_n^\circ)[F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ)e + o_p(1) . \end{aligned}$$

We next show that

$$1/s^2 = (n-p)/e'(I - P_F)e + o_p(1/n)$$

where

$$P_F = F(\theta_n^\circ)[F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ) .$$

Fix a realization of the errors  $\{e_t\}$  for which  $\lim_{n \rightarrow \infty} s^2 = \sigma^2$  and

$\lim_{n \rightarrow \infty} e'(I - P_F)e/(n-p) = \sigma^2$ ; almost every realization is such (Problem 2).

Choose  $N$  so that if  $n > N$  then  $s^2 > 0$  and  $e'(I - P_F)e > 0$ . Using

$$s^2 = e'(I - P_F)e/(n-p) + o_p(1/n)$$

and Taylor's theorem we have

$$1/s^2 = (n-p)/e'(I - P_F)e - [(n-p)/e'(I - P_F)e]^2 o_p(1/n) .$$

The term  $[(n-p)/e'(I - P_F)e]^2$  is bounded for  $n > N$  because  
 $\lim_{n \rightarrow \infty} [(n-p)/e'(I - P_F)e]^2 = 1/\sigma^4$ . One concludes that  
 $1/s^2 = (n-p)/e'(I - P_F)e + o_p(1/n)$  which completes the argument.

The next task is to show that if the errors are normally distributed then

$$W = Y + o_p(1)$$

where

$$Y \sim F'(q, n-p, \lambda)$$

$$\lambda = h'(\theta_n^0) \{ H(\theta_n^0) [F'(\theta_n^0) F(\theta_n^0)]^{-1} H'(\theta_n^0) \}^{-1} h(\theta_n^0) / (2\sigma^2) .$$

Now

$$W = n h'(\hat{\theta}_n) \{ \hat{H}[(1/n)F'(\hat{\theta}_n)F(\hat{\theta}_n)]^{-1} \hat{H}' \}^{-1} h(\hat{\theta}_n) / (qs^2)$$

and as notation write

$$\begin{aligned} \sqrt{n} h(\hat{\theta}_n) &= \sqrt{n} h(\theta_n^0) + \sqrt{n} H(\theta_n^0) [F'(\theta_n^0) F(\theta_n^0)]^{-1} F'(\theta_n^0) e + o_p(1) \\ &= \mu + U + o_p(1) \end{aligned}$$

$$\begin{aligned} \{ \hat{H}[(1/n)F'(\hat{\theta}_n)F(\hat{\theta}_n)]^{-1} \hat{H}' \}^{-1} &= \{ H(\theta_n^0) [(1/n)F'(\theta_n^0)F(\theta_n^0)]^{-1} H'(\theta_n^0) \}^{-1} + o_p(1) \\ &= A^{-1} + o_p(1) \end{aligned}$$

whence

$$\begin{aligned} W &= [\mu + U + o_p(1)]' A^{-1} [\mu + U + o_p(1)] [(n-p)/e'(I - P_F)e + o_p(1)] / q \\ &= \frac{(\mu + U)' A^{-1} (\mu + U) / (qs^2)}{e'(I - P_F)e / [\sigma^2(n-p)]} + o_p(1) \\ &= Y + o_p(1) . \end{aligned}$$

Assuming normal errors then

$$U \sim N_q(0, \sigma^2 A)$$

which implies that (Appendix 1)

$$(\mu + U)'A^{-1}(\mu + U)/\sigma^2 \sim \chi^2(q, \lambda)$$

with

$$\begin{aligned} \lambda &= \mu'A^{-1}\mu/(2\sigma^2) \\ &= n h'(\theta_n^\circ) \{H(\theta_n^\circ) [(1/n)F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}H'(\theta_n^\circ)\}^{-1} h(\theta_n^\circ)/(2\sigma^2) . \end{aligned}$$

Since  $A(I - P_F) = 0$ ,  $U$  and  $(I - P_F)e$  are independently distributed whence  $(\mu + U)'A^{-1}(\mu + U)$  and  $e'(I - P_F)e = e'(I - P_F)'(I - P_F)e$  are independently distributed. This implies that  $Y \sim F'(q, n-p, \lambda)$  which completes the argument.

Simply by rescaling  $s^2$  in the foregoing we have that

$$\begin{aligned} (\text{SSE}_{\text{full}})/n &= e'P_F^\perp e/n + o_p(1/n) \\ n/(\text{SSE}_{\text{full}}) &= n/e'P_F^\perp e + o_p(1/n) \end{aligned}$$

where

$$P_F^\perp = I - P_F = I - F(\theta_n^\circ)[F'(\theta_n^\circ)F(\theta_n^\circ)]^{-1}F'(\theta_n^\circ) ;$$

recall that

$$\begin{aligned} \text{SSE}_{\text{full}} &= \|y - f(\hat{\theta}_n)\|^2 \\ \text{SSE}_{\text{reduced}} &= \|y - f(\tilde{\theta}_n)\|^2 = \|y - f[g(\hat{\rho}_n)]\|^2 . \end{aligned}$$

The claim that

$$(\text{SSE}_{\text{reduced}})/n = (e + \delta)'P_{FG}^\perp(e + \delta)/n + o_p(1/n)$$

with

$$\begin{aligned} \delta &= f(\theta_n^\circ) - f(\theta_n^*) = f(\theta_n^\circ) - f[g(\rho_n^\circ)] \\ P_{FG}^\perp &= I - P_{FG} = I - F(\theta_n^\circ)G(\rho_n^\circ)[G'(\rho_n^\circ)F'(\theta_n^\circ)F(\theta_n^\circ)G(\rho_n^\circ)]^{-1}G'(\rho_n^\circ)F'(\theta_n^\circ) \end{aligned}$$

comes fairly close to being a restatement of a few lines of the proof of

Theorem 13 of Chapter 3. In that proof we find the equations

$$\bar{H} \sqrt{n} (\tilde{\theta}_n - \theta_n^*) = o_s(1)$$

$$\sqrt{n} (\tilde{\theta}_n - \theta_n^*) = \bar{g}^{-1} \sqrt{n} (\partial/\partial \theta)_{s_n}(\tilde{\theta}_n) - \bar{g}^{-1} \sqrt{n} (\partial/\partial \theta)_{s_n}(\theta_n^*) + o_s(1)$$

which, using arguments that have become repetitive at this point, can be rewritten as

$$H \sqrt{n} (\tilde{\theta}_n - \theta_n^*) = o_s(1)$$

$$\sqrt{n} (\tilde{\theta}_n - \theta_n^*) = g^{-1} [\sqrt{n} (\partial/\partial \theta)_{s_n}(\tilde{\theta}_n) - \sqrt{n} (\partial/\partial \theta)_{s_n}(\theta_n^*)] + o_p(1)$$

with  $g = g_n^0$  and  $H = H(\theta_n^*)$ . Using the conclusion of Theorem 13 of Chapter 3 one can substitute for  $\sqrt{n} (\partial/\partial \theta)_{s_n}(\tilde{\theta}_n)$  to obtain

$$\sqrt{n} [(\partial/\partial \theta)_{s_n}(\tilde{\theta}_n)]' \sqrt{n} (\tilde{\theta}_n - \theta_n^*) = o_p(1)$$

$$\sqrt{n} (\tilde{\theta}_n - \theta_n^*) = -g^{-1} [g - H'(Hg^{-1}H')^{-1}H]g^{-1} \sqrt{n} (\partial/\partial \theta)_{s_n}(\theta_n^*) + o_p(1).$$

Then using Taylor's theorem

$$\begin{aligned} n[s_n(\tilde{\theta}_n) - s_n(\theta_n^*)] &= -n[(\partial/\partial \theta)_{s_n}(\tilde{\theta}_n)](\tilde{\theta}_n - \theta_n^*) - (n/2)(\tilde{\theta}_n - \theta_n^*)' [g + o_s(1)](\tilde{\theta}_n - \theta_n^*) \\ &= (-n/2)(\tilde{\theta}_n - \theta_n^*)' g(\tilde{\theta}_n - \theta_n^*) + o_p(1) \\ &= (-n/2)[(\partial/\partial \theta)_{s_n}(\theta_n^*)]' [g^{-1} - g^{-1}H(Hg^{-1}H')^{-1}Hg^{-1}] [(\partial/\partial \theta)_{s_n}(\theta_n^*)]. \end{aligned}$$

Using the identity obtained in Section 6 of Chapter 3 we have

$$g^{-1} - g^{-1}H(Hg^{-1}H')^{-1}Hg^{-1} = G(G'gG)^{-1}G'$$

whence

$$n s_n(\tilde{\theta}_n) = n s_n(\theta_n^*) - (n/2)[(\partial/\partial\theta)s_n(\theta_n^*)]'G(G'G)^{-1}G'[(\partial/\partial\theta)s_n(\theta_n^*)] + o_p(1)$$

Using Taylor's theorem, the Uniform Strong Law, and the Pitman drift assumption we have

$$\begin{aligned} (\partial/\partial\theta)s_n(\theta_n^*) &= (-2/n)\sum_{t=1}^n [e_t + f(x_t, \theta_n^\circ) - f(x_t, \theta_n^*)](\partial/\partial\theta)f(x_t, \theta_n^*) \\ &= (-2/n)\sum_{t=1}^n [e_t + f(x_t, \theta_n^\circ) - f(x_t, \theta_n^*)](\partial/\partial\theta)f(x_t, \theta_n^\circ) \\ &\quad + (1/\sqrt{n})(-2/n)\sum_{t=1}^n [e_t + f(x_t, \theta_n^\circ) - f(x_t, \theta_n^*)] \\ &\quad \times \begin{pmatrix} (\partial/\partial\theta')(\partial/\partial\theta_1)f(x_t, \bar{\theta}_{1n}) \\ \vdots \\ (\partial/\partial\theta')(\partial/\partial\theta_p)f(x_t, \bar{\theta}_{pn}) \end{pmatrix} \sqrt{n} (\theta_n^\circ - \theta_n^*) \\ &= (-2/n) F'(\theta_n^\circ)(e + \delta) + o_p(1/\sqrt{n}) . \end{aligned}$$

Substitution and algebraic reduction yields (Problem 3)

$$n s_n(\tilde{\theta}_n) = (e + \delta)'(e + \delta) - (e + \delta)'P_{FG}(e + \delta) + o_p(1)$$

which proves the claim.

The following are the characterizations used in Chapter 1 that have not yet been verified

$$(\text{SSE}_{\text{reduced}})/(\text{SSE}_{\text{full}}) = (e + \delta)' P_{FG}^{\perp} (e + \delta) / e' P_F^{\perp} e = o_p(1/n)$$

$$\tilde{D}' (\tilde{F}' \tilde{F})^{-1} \tilde{D} / n = (e + \delta)' (P_F - P_{FG}) (e + \delta) / n + o_p(1/n)$$

$$\frac{\tilde{D}' (\tilde{F}' \tilde{F}) \tilde{D} / q}{\text{SSE}(\tilde{\theta}) / (n-p)} = \frac{(e + \delta)' (P_F - P_{FG}) (e + \delta) / q}{e' (I - P_F) e / (n-p)} + o_p(1)$$

$$\frac{n \tilde{D}' (\tilde{F}' \tilde{F}) \tilde{D}}{\text{SSE}(\tilde{\theta})} = \frac{n(e + \delta)' (P_F - P_{FG}) (e + \delta)}{(e + \delta)' (I - P_{FG}) (e + \delta)} + o_p(1) .$$

Except for the second, these are obvious at sight. Let us sketch the verification of the second characterization

$$\begin{aligned} \tilde{D}' \tilde{F}' \tilde{F} \tilde{D} &= [y - f(\tilde{\theta}_n)]' \tilde{F}' (\tilde{F}' \tilde{F})^{-1} \tilde{F}' [y - f(\tilde{\theta}_n)] \\ &= (n/4) [(\partial/\partial \theta) s_n(\tilde{\theta}_n)]' [(1/n) \tilde{F}' \tilde{F}]^{-1} [(\partial/\partial \theta) s_n(\tilde{\theta}_n)] \\ &= (n/2) [(\partial/\partial \theta) s_n(\tilde{\theta}_n)]' [\mathcal{J} + o_s(1)]^{-1} [(\partial/\partial \theta) s_n(\tilde{\theta}_n)] \\ &= (n/2) [(\partial/\partial \theta) s_n(\theta_n^*)]' \mathcal{J}^{-1} H' (H \mathcal{J}^{-1} H')^{-1} H \mathcal{J}^{-1} [(\partial/\partial \theta) s_n(\theta_n^*)] + o_p(1) \\ &= (n/2) [(\partial/\partial \theta) s_n(\theta_n^*)]' [\mathcal{J}^{-1} - G(G' \mathcal{J} G)^{-1} G'] [(\partial/\partial \theta) s_n(\theta_n^*)] + o_p(1) \\ &= (1/n) (e + \delta)' F(\theta_n^o) [(Q_n^o)^{-1} - G(G' Q_n^o G)^{-1} G'] F'(\theta_n^o) (e + \delta) + o_p(1) \\ &= (1/n) (e + \delta)' F(\theta_n^o) [(Q_n^o)^{-1} - G(G' Q_n^o G)^{-1} G'] F'(\theta_n^o) (e + \delta) + o_p(1) \\ &= (e + \delta)' (P_F - P_{FG}) (e + \delta) + o_p(1) . \end{aligned}$$

## PROBLEMS

1. Give a detailed derivation of the four characterizations listed in the preceding paragraph.

2. Cite the theorem which permits one to claim that  $\lim_{n \rightarrow \infty} s^2 = \sigma^2$  almost surely and prove directly that  $\lim_{n \rightarrow \infty} e'(I - P_F)e/(n-p) = \sigma^2$  almost surely.

3. Show in detail that  $(\partial/\partial\theta)s_n(\theta_n^*) = (-2/n)F'(\theta_n^0)(e+\delta) + o_p(1/\sqrt{n})$  suffices to reduce  $(n/2)[(\partial/\partial\theta)s_n(\theta_n^*)]'G(G'G)^{-1}G'[(\partial/\partial\theta)s_n(\theta_n^*)]$  to  $(e+\delta)'P_{FG}(e+\delta)$ .

## 4. REFERENCES

Callant, A. Ronald and Alberto Holly (1980), "Statistical Inference in an Implicit, Nonlinear, Simultaneous Equation Model in the Context of Maximum Likelihood Estimation," *Econometrica* 48, 697-720.

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